Lecture 23: Whitehead's Theorem

Whitehead's Thm: $f: X \to Y$ a map between connected CW complexes. If f_* is an $\cong$ on all π_n then f is a homotopy equivalence. Moreover, f is the inclusion of a subcomplex, then Y deformation retracts to X.

Compression Lemma: (X, A) a CW pair, $(Y, \overset{\neq\emptyset}{B})$ any pair. For each n where $X \setminus A$ has n-cell, assume $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f: (X, A) \to (Y, B)$ is homotopic, rel A, to a map $f: X \to B$.

Pf of Lemma: Inductively, build $f_k: (X, A) \to (Y, B)$ so that
① f_k is homotopic to f, rel A.
② $f_k(X^k \cup A) \subseteq B$.
③ homotopy from f_{k-1} to f_k is const on $X^{k-1} \cup A$.

[f_{k+1} build from f_k by first changing on X^k, then applying the homotopy extension thm.]

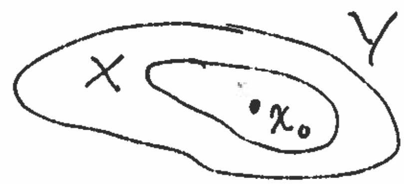
Now define $f_\infty: (X, A) \to (Y, B)$ by

$$f_\infty\big|_{X^k} = f_k\big|_{X^k} = f_j\big|_{X^k} \quad \text{for } j \geq k.$$

Then f_∞ is homotopic to f via a homotopy that implements the homotopy from f_k to f_{k+1} in the time interval $[1-2^{-k}, 1-2^{-(k+1)}]$. $\blacksquare$

Works since X has the weak topology: $U \subseteq X$ is open $\iff$ $U \cap X^k$ is open for each k.

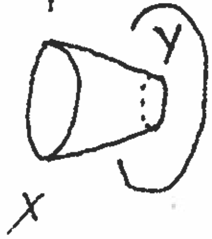
<u>Pf of Thm:</u> Suppose X is a subcomplex and f is inclusion. Long exact sequence gives (for any $x_0 \in X$)



$$\begin{tikzcd}[column sep=small]
{} \arrow[r] & \pi_n X \arrow[r, "f_*", "\cong"'] & \pi_n Y \arrow[rr] \arrow[dr] & & \pi_n(Y,X) \arrow[rr] \arrow[dr] & & \pi_{n-1}(X) \arrow[r, "\cong"'] & \pi_{n-1}(Y) \arrow[r] & {} \\
& & & 0 \arrow[ur] & & 0 \arrow[ur] & & &
\end{tikzcd}$$

and so $\pi_n(Y,X) = 0$. Applying the Compression Lemma to $id : (Y,X) \to (Y,X)$ gives the needed deformation retraction

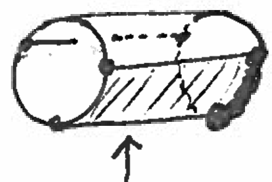
If $f: X \to Y$ is any map, consider the

<u>mapping cylinder</u> $M_f =$  $= \dfrac{X \times [0,1] \amalg Y}{(x,1) \sim f(x)}$

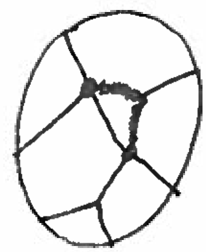
<u>Note</u>: M_f def retracts to Y; $X \hookrightarrow M_f \xrightarrow[\simeq]{\text{retract}} Y$, with the composite $X \to Y$ equal to f

<u>Claim</u>: When f_* is an $\cong$ on π_n, then M_f deformation retracts to X. [$\Rightarrow X \hookrightarrow M_f$ is a homotopy equiv, hence $X \xrightarrow{f} Y$ is.]

If f is <u>cellular</u>, i.e. $f(X^k) \subseteq Y^k$ for all k, then M_f is a CW complex with $X \times \{0\}$ as a subcomplex, allowing us to apply the earlier case. So have reduced Whitehead's Thm to:



(cell in X) $\times I$



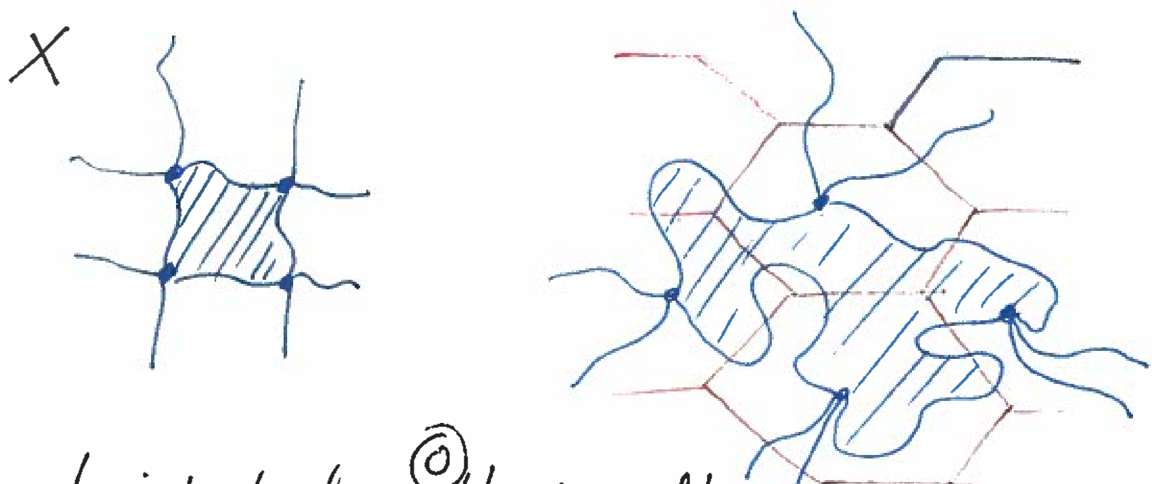
<u>Thm</u>: Every $f: X \to Y$ of CW complexes is homotopic to a cellular map. If f is already cellular on a subcomplex A of X, then the homotopy is const on A.

<u>Cor</u>: $\pi_n(S^k) = 0$ for $n < k$.

Pf of Cor: Consider the usual cell decomp of S^n and S^k with one zero cell (the base pt) and one other cell. Then the Thm implies that any map $S^n \to S^k$ is homotopic (rel base pt) to the const map. ▨

[Query: how did you show $\pi_1(S^2) = 1$?]

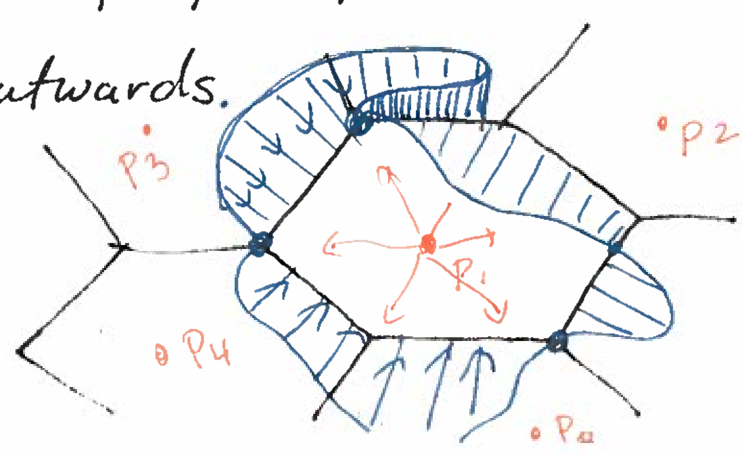
Pf of Cellular Approximation: Idea



Proceed inductively: ⓪ Homotope $f|_{X^0}$ so that $f(X^0) \subseteq Y^0$; extend to all of X by homotopy extension thm.

① Homotope $f: X^1 \to Y$, rel X^0, so that $f(X^1) \subseteq Y^1$ by picking p_i in each cell of $Y \setminus Y^1$ not in image and pushing outwards.

Again expend to all of X by homotopy ext. thm.



(n) Repeat for each n in the same manner.

[Query: What is missing here?]

<u>Lemma</u>: $Z = \overset{\text{some sp}}{W} \cup (k\text{-cell } e^k)$. For $n < k$, and map $f: I^n \to Z$ is homotopic, rel $f^{-1}(W)$, to a map where $g(I^n)$ is a <u>proper subset</u> of $\operatorname{int}(e^k)$.

<u>Lemma</u>: Any cpt A in a CW complex X meets the interiors of finitely many cells.