Lecture 5: The cup product.

Setting: X space, R a ring $[\mathbb{Z}, \mathbb{Q}, \mathbb{Z}/p\mathbb{Z}, ..]$

H^* cohom with R-coeff.

Cup product:

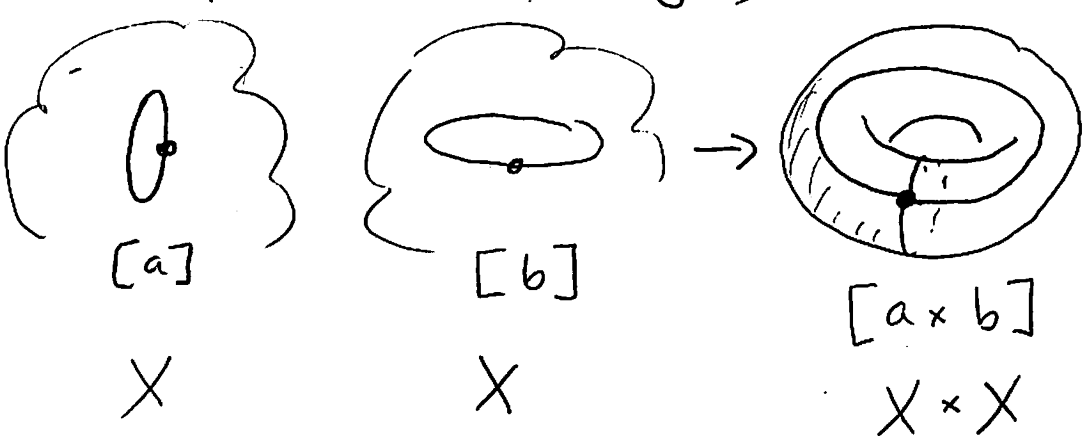
$$H^k(X) \times H^l(X) \xrightarrow[\cup]{} H^{k+l}(X)$$

making $H^*(X) = \bigoplus_k H^k(X)$ into a (noncommutative) ring.

[Several (equivalent) ways of thinking about.]

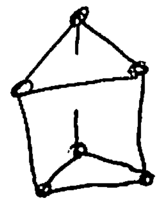
X a CW complex $\rightsquigarrow$ $\underbrace{X \times X}_{\text{cell str w/ prod. of cells}}$ also a CW cplx.

Cross product (homology)



For cocycles $\varphi \in C^k(X)$ and $\psi \in C^l(X)$, define

$$(\varphi \times \psi)(\sigma \times \tau) = \begin{cases} \varphi(\sigma)\psi(\tau) & \text{if } \dim \sigma = k \text{ and } \dim \tau = l \\ 0 & \text{otherwise} \end{cases}$$



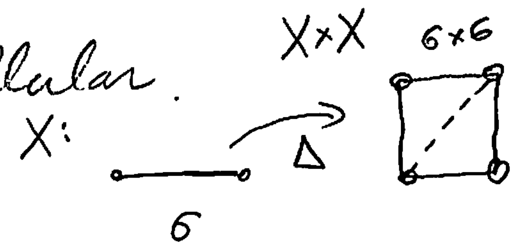
to get $[\varphi \times \psi] \in H^{l+k}(X \times X)$.

Consider $\Delta : X \to X \times X$
$\quad x \mapsto (x,x)$

Cup product $[\varphi] \cup [\psi] = \Delta^*([\varphi \times \psi])$

$$\begin{array}{ccccc} H^k(X) \oplus H^l(X) & \longrightarrow & H^{k+l}(X \times X) & \xrightarrow{\Delta^*} & H^{k+l}(X) \\ ([\varphi],[\psi]) & & [\varphi \times \psi] & & \end{array}$$

<u>Technical issue</u>: Δ is not cellular.



<u>Workarounds</u> (§3.B)
- homotope Δ to a cellular map
- Define $\times$ on <u>singular</u> cohom.

<u>Note</u>: Doesn't give a prod. in hom. as Δ_* goes the other way.

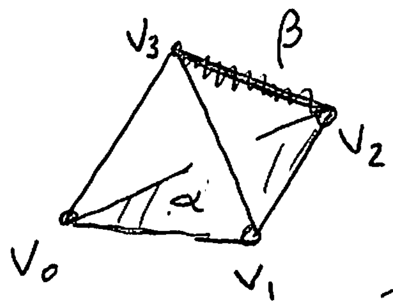
Work with singular hom: $C_n(X;R) =$ Formal R-sums of $\sigma : \Delta_n \to X$

Define $C^k(X) \times C^\ell(X) \longrightarrow C^{k+\ell}(X)$ [Coeff in R.]

$$\alpha \quad \beta \longmapsto \alpha \cup \beta$$

by $(\alpha \cup \beta)(\sigma) = \alpha(\overbrace{\sigma|_{[v_0, \dots, v_k]}}^{\text{front}})\, \beta(\overbrace{\sigma|_{[v_k, \dots, v_{k+\ell}]}}^{\text{back}})$

$k = 2 \quad \ell = 1$ where $\sigma : \Delta^{k+\ell} \to X$ is in $C^{k+l}(X)$.



<u>Lemma</u> $\delta(\alpha \cup \beta) = \delta\alpha \cup \beta + (-1)^k \alpha \cup \delta\beta$

<u>Pf</u>: Apply both sides to a $k+\ell+1$ simplex σ and calculate. ▨

<u>Cor</u>: Get an induced map

$$H^k(X) \times H^\ell(X) \longrightarrow H^{k+\ell}(X)$$
$$[\alpha] \times [\beta] \longmapsto [\alpha \cup \beta]$$

bilinear, associative, and distributive (follows from chain level)

<u>Pf</u>: Have $\delta(\alpha \cup \beta) = 0$ and if $\alpha' = \alpha + \delta\varphi$ then

$$\alpha' \cup \beta = \alpha \cup \beta + (\delta\varphi) \cup \beta = \alpha \cup \beta + \delta(\varphi \cup \beta).$$

as $\cup$ is bilinear on cochains

by lemma and $\delta\beta = 0$ ▨

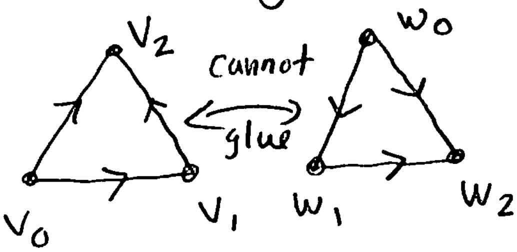
Thm [Next time] $[\alpha]\times[\beta] = (-1)^{kl}[\beta]\cup[\alpha]$

If R has a mult unit 1_R then so does $H^*(X)$:

$1_{H^*} \in H^0(X)$ defined by $1_{H^*}(0\text{-simplex}) = 1_R$.

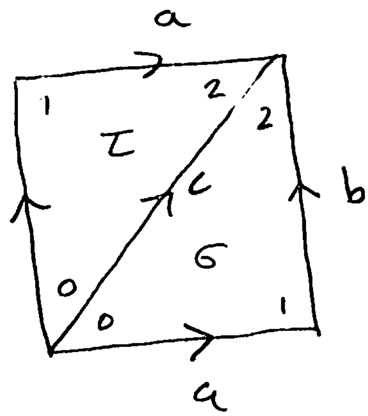
[Can't compute directly from a CW complex, e.g. $\mathbb{C}P^2$ vs. $S^2 \vee S^4$ have same CW-chains but diff. cup products. Instead need Δ-complexes]

Recall: In a Δ-complex, the order of the verts of each tet agrees with each subsimplex.



Ex: T^2

$R = \mathbb{Z}$



$H_1(T) = \mathbb{Z}^2$ gen by $[a], [b]$

$[c] = [a] + [b]$

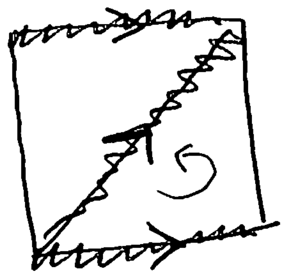
$H_2(T) = \mathbb{Z}$ gen by $[\sigma - \tau]$

U.T.C says: $H^1(T) = \mathbb{Z}^2$ gen by $[\alpha], [\beta]$

with $[\alpha](a) = 1 \qquad [\beta](a) = 0$

$\quad\;\; [\alpha](b) = 0 \qquad [\beta](b) = 1.$

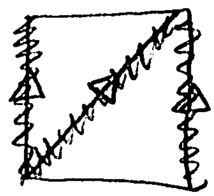
Concretely: $\alpha = \chi_a + \chi_c$ where e.g. χ_a is 1 on a and zero on the other vertices.



α

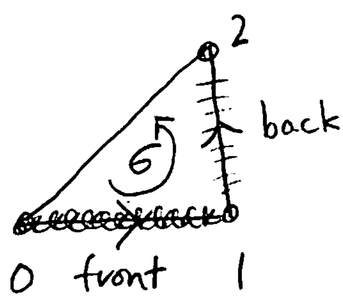
and

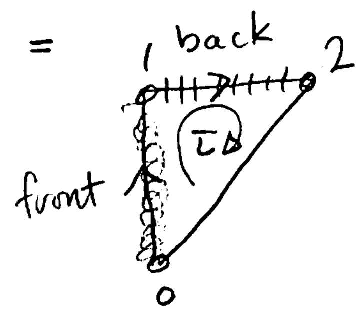
$\beta = \chi_b + \chi_c$



Also $H^2(T) \cong \mathbb{Z} = [\chi_\sigma]$ since $h([\chi_\sigma])([\sigma - \tau]) = 1$.

Now $\alpha \cup \beta(\sigma) = \alpha(\text{front}) \cdot \beta(\text{back}) = 1 \cdot 1 = 1$





and $\alpha \cup \beta(\tau) = \alpha(b) \cdot \beta(a) = 0 \cdot 0 = 0$.

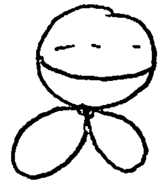
So $\alpha \cup \beta = \chi_\sigma \Rightarrow [\alpha] \cup [\beta] = [\chi_\sigma]$

Can check: $\alpha \cup \alpha = \beta \cup \beta = 0$ (on chains)

$\beta \cup \alpha = \chi_\tau$

$\Rightarrow [\beta] \cup [\alpha] = -[\chi_\sigma]$ since $\delta \chi_a = \chi_\sigma + \chi_\tau$.

Ex: $X = S^2 \vee S^1 \vee S^1$



has same cohom as T^2 but $[\alpha] \cup [\beta] = 0$.

[Note: If use CW complexes get same chains for both!]

Add'l props:

Functoriality: $f : X \to Y$ cont.

$$f^* : H^*(Y) \to H^*(X)$$

is a ring hom., that is $f^*(\alpha \cup \beta)$

$= f^*(\alpha) \cup f^*(\beta)$

Relative version: $C^n(X, A; R) = \varphi : C_n(X) \to R$ that vanish on $C_n(A)$

$$\begin{tikzcd}
 & H^k(X,A) \times H^\ell(X,A) \arrow[dl] \arrow[dr] \arrow[d, "\cup"] & \\
H^k(X) \times H^\ell(X,A) \arrow[r, "\cup"'] & H^{k+\ell}(X,A) & H^k(X,A) \times H^\ell(X) \arrow[l, "\cup"]
\end{tikzcd}$$