Lecture 7: Cohomology of products

HW #2: Due Wed Sept 22
Hatcher 3.2: # 7, 11, 13; 3.3: # 1, 8.

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What is $H^*(X\times Y)$? Starting pt:

$$\begin{array}{ccc} H^*(X) \times H^*(Y) & \xrightarrow{\times} & H^*(X\times Y) \\ \alpha \quad\quad \beta & \longmapsto & \alpha\times\beta = p_X^*(\alpha)\cup p_Y^*(\beta) \end{array}$$

[Might hope this is an isom, but...]

$X = S^1 \quad Y = \{pt\} \qquad\qquad X\times Y = S^1$

$$(\mathbb{Z}_{(0)}\oplus\mathbb{Z}_{(1)})\oplus\mathbb{Z}_{(0)} \xrightarrow{\times} \mathbb{Z}_{(0)}\oplus\mathbb{Z}_{(1)} \qquad \text{Nope!}$$

Also, $\times$ is bilinear, not a homomorphism. That is

$$(\alpha_1+\alpha_2)\times\beta = \alpha_1\times\beta + \alpha_2\times\beta \quad \text{and the reverse}$$

which means

$$\times\big(\underbrace{(\alpha_1,\beta_1)+(\alpha_2,\beta_2)}_{(\alpha_1+\alpha_2,\ \beta_1\times\beta_2)}\big) = \alpha_1\times\beta_1 + \alpha_1\times\beta_2 + \alpha_2\times\beta_1 + \alpha_2\times\beta_2$$

$$\neq \times(\alpha_1,\beta_1) + \times(\alpha_2,\beta_2).$$

unless $\alpha_1\times\beta_2 + \alpha_2\times\beta_1 = 0$.

Solution: Replace $\times$ by $\otimes$.

Tensor Product: R ring A, B are R-modules.

$$A \otimes_R B = \bigoplus_{(a,b) \in A \times B} R[a \otimes b] \Big/ \begin{array}{l} (a+a') \otimes b = a \otimes b + a' \otimes b \\ a \otimes (b+b') = a \otimes b + a \otimes b' \\ (ra) \otimes b = a \otimes (rb) \end{array}$$

Ex: $R = \mathbb{Q}$, $A = \mathbb{Q}^2$ with basis $\{a_1, a_2\}$
$B = \mathbb{Q}^3$ with basis $\{b_1, b_2, b_3\}$

Then $A \otimes_{\mathbb{Q}} B \cong \mathbb{Q}^6$ with basis $\{a_i \otimes b_j\}$.

Typical elt $3a_1 \otimes b_2 + 4a_2 \otimes b_3 - 6a_3 \otimes b_1$

Idea $(3a_1 - 2a_2) \otimes (b_1 + 6b_3) = 3a_1 \otimes b_1 + 18 a_1 \otimes b_3 - 2a_2 \otimes b_1 - 12 a_2 \otimes b_3$

Key: An R-bilinear map $\varphi : A \times B \to C$ gives
a homomorphism $A \otimes_R B \to C$
$a \otimes b \longmapsto \varphi(a,b)$

Conversely, a homom. $\psi : A \otimes_R B \to C$ gives
a bilinear map $A \times B \to C$
$(a, b) \longmapsto \psi(a \otimes b)$.

[$A \otimes B$ is the "smallest" and so "universal" thingy for which this is true. See last page for the precise statement.]

Ex: For any R, have $R^a \otimes_R R^b \cong R^{a \cdot b}$ and $R \otimes_R B \cong B$.

Note: Any abelian gp is a $\mathbb{Z}$-module

$$r \cdot a := \sum_{i=1}^{r} a \quad \text{for } a \in A.$$

For abelian gps A and B, one writes $A \otimes B$ for $A \otimes_{\mathbb{Z}} B$.

[Reminder: $\otimes$ shows up in UCT for homology.]

Ex: $\mathbb{Z}/n \otimes B \cong B/nB \Rightarrow \mathbb{Z}/3 \otimes \mathbb{Z}/5 = 0$.

$\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$ but $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{R}^4$

$$\begin{array}{ccc} \alpha \otimes \beta & \longmapsto & \alpha \times \beta \\ H^*(X) \otimes_R H^*(Y) & \xrightarrow[\times]{} & H^*(X \times Y) \end{array}$$

Upshot: Get a homom $H^*(X) \otimes_R H^*(Y) \xrightarrow[\times]{} H^*(X \times Y)$ where we are using R-coeffs.

Define a mult on $H^*(X) \otimes_R H^*(Y)$ by:

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{|b||c|} (a \cup c) \otimes (b \cup d)$$

where a, b, c, d are "pure" (in some $H^k(X)$ or $H^l(Y)$) and $|b|, |c|$ denote the cor. degree

Künneth Thm: If X and Y are CW complexes and every $H^k(Y)$ is a finitely gen free R-module, then $H^*(X) \otimes_R H^*(Y) \xrightarrow[\times]{} H^*(X \times Y)$ is an isomorphism of rings.

Cor: $H^*(T^n; R)$ is the exterior algebra on $\alpha_1, \dots, \alpha_n$ where $\alpha_i = p_i^*(\alpha)$ for $p_i : T^n \to S^1$ the ith proj and α is a gen for $H^1(S^1; R)$.

[Pf: Induct on n; same idea gives a Künneth formula for $X \times Y \times Z$ or even $X_1 \times X_2 \times X_3 \times \cdots \times X_n$]

Cor: In the setting the Künneth thm applies, we have $H^n(X \times Y) \cong \bigoplus_{0 \le k \le n} H^k(X) \otimes_R H^{n-k}(Y)$.

Note: Modules over a field F are just vector spaces, and hence free. So for fields, the Künneth Thm applies for any finite CW complexes.

[Also versions for homology and when $H^*(Y)$ is not free...]

§3.B: X, Y are CW complexes and R is a PID (e.g. $\mathbb{Z}$, a field) then there is a nat'l short exact seq

$$0 \to \bigoplus_i H_i(X) \otimes_R H_{n-i}(Y) \xrightarrow{\times} H_n(X \times Y) \longrightarrow \bigoplus_i \mathrm{Tor}_R(H_i(X), H_{n-i-1}(Y)) \to 0.$$

which splits (but not nat'ly).

Aside: Univ. char. $A \times B \underset{R\text{-bilinear}}{\xrightarrow{\ m\ }} M$ ← R-module

Such that $\forall$ R-bilinear $A \times B \longrightarrow C$
$\exists!$ R-module hom $M \longrightarrow C$

where

$$\begin{tikzcd} & M \arrow[dd, dashed] \\ A \times B \arrow[ur] \arrow[dr] & \\ & C \end{tikzcd}$$

commutes.