What is $H^*(X \times Y)$? Starting pt:

$$H^*(X) \times H^*(Y) \to H^*(X \times Y)$$

$$\alpha \quad \beta \quad \to \quad \alpha \times \beta = P_X^*(\alpha) \cup P_Y^*(\beta)$$

[Might hope this is an isom, but...]

$X = S^1$, $Y = \text{pt}$

$X \times Y = S^1$

\[
(Z_{(1)} \oplus Z_{(1)}) \oplus Z_{(1)} \to Z_{(0)} \oplus Z_{(1)}
\]

Nope!

Also, $x$ is bilinear, not a homomorphism. That is

$$\left( \alpha_1 + \alpha_2 \right) \times \beta = \alpha_1 \times \beta + \alpha_2 \times \beta$$

and the reverse, which means

$$x \left( \left( \alpha_1, \beta_1 \right) + \left( \alpha_2, \beta_2 \right) \right) = \alpha_1 \times \beta_1 + \alpha_1 \times \beta_2 + \alpha_2 \times \beta_1 + \alpha_2 \times \beta_2$$

$$\left( \left( \alpha_1 + \alpha_2, \beta_1 \times \beta_2 \right) \right) \neq x(\alpha_1, \beta_1) + x(\alpha_2, \beta_2).$$

unless $\alpha_1 \times \beta_2 + \alpha_2 \times \beta_1 = 0$. 
Solution: Replace $x$ by $\otimes$.

Tensor Product: $R$ ring, $A, B$ are $R$-modules.

$$A \otimes_R B = \bigoplus R[a \otimes b] \quad \begin{aligned}
(a + a') \otimes b &= a \otimes b + a' \otimes b \\
(a \otimes (b + b')) &= a \otimes b + a \otimes b' \\
(r a) \otimes b &= a \otimes (r b)
\end{aligned}$$

Ex: $R = \mathbb{Q}$, $A = \mathbb{Q}^2$ with basis $\{a_1, a_2\}$
$B = \mathbb{Q}^3$ with basis $\{b_1, b_2, b_3\}$

Then $A \otimes \mathbb{Q} B \cong \mathbb{Q}^6$ with basis $\{a_i \otimes b_j\}$.

Typical elt: $3a_1 \otimes b_2 + 4a_2 \otimes b_3 - 6a_3 \otimes b_1$

Idea: $(3a_1 - 2a_2) \otimes (b_1 + 6b_3) = 3a_1 \otimes b_1 + 18a_1 \otimes b_3 - 2a_2 \otimes b_1 - 12a_2 \otimes b_3$

Key: An $R$-bilinear map $\varphi: A \times B \to C$ gives
a homomorphism $A \otimes_R B \to C$

$$a \otimes b_1 \mapsto \varphi(a, b_1)$$

Conversely, a homom. $\psi: A \otimes_R B \to C$ gives
a bilinear map $A \times B \to C$

$$(a, b) \mapsto \psi(a \otimes b)$$

$A \otimes B$ is the "smallest" and so "universal" thing for which this is true. See last page for the precise statement.
Ex: For any $R$, have $R^a \otimes_R R^b \cong R^{a \cdot b}$ and $R \otimes_R B \cong B$.

Note: Any abelian gp is a $\mathbb{Z}$-module
\[ r \cdot a = \sum_{i=1}^{r} a \] for $a \in A$.

For abelian gps $A$ and $B$, one writes $A \otimes B$ for $A \otimes_{\mathbb{Z}} B$.

[Reminder: $\otimes$ shows up in UCT for homology.]

Ex: $\mathbb{Z}/n \otimes \mathbb{Z}/B \cong \mathbb{Z}/nB \Rightarrow \mathbb{Z}/3 \otimes \mathbb{Z}/5 = 0$.

$C \otimes_{\mathbb{Z}} C \cong C$ but $C \otimes_{\mathbb{R}} C \cong \mathbb{R}^4$

\[ \alpha \otimes \beta \rightarrow \alpha \times \beta \]

Upshot: Get a homom $H^*(X) \otimes_R H^*(Y) \rightarrow H^*(X \times Y)$ where we are using $R$-coeffs.

Define a mult on $H^*(X) \otimes_R H^*(Y)$ by:

\[ (a \otimes b) \cdot (c \otimes d) = (-1)^{|b| |c|} (a \cup c) \otimes (b \cup d) \]

where $a, b, c, d$ are "pure" (in some $H^k(X)$ or $H^l(Y)$) and $|b|, |c|$ denote the cor. degree.
Künneth Thm: If $X$ and $Y$ are CW complexes and every $H^k(Y)$ is a finitely gen free $R$-module, then $H^*(X) \otimes_R H^*(Y) \xrightarrow{x} H^*(X \times Y)$ is an isomorphism of rings.

Cor: $H^*(T^n; R)$ is the exterior algebra on $\alpha_1, \ldots, \alpha_n$ where $\alpha_i = p_i^*(\alpha)$ for $p_i: T^n \to S^1$ the $i$th proj and $\alpha$ is a gen for $H^*(S^1; R)$.

[Proof: Induct on $n$; same idea gives a Künneth formula for $X \times Y \times Z$ or even $X_1 \times X_2 \times X_3 \times \cdots \times X_n$]

Cor: In the setting the Künneth thm applies, we have $H^n(X \times Y) \cong \bigoplus_{\text{osksh}} H^k(X) \otimes_R H^{n-k}(Y)$.

Note: Modules over a field $F$ are just vector spaces, and hence free. So for fields, the Künneth Thm applies for any finite CW complexes.
§3.B: $X, Y$ are CW complexes and $R$ is a PID (e.g. $\mathbb{Z}$, a field) then there is a natural short exact seq

$$0 \to \bigoplus_i H_i(X) \otimes_R H_{n-i}(Y) \xrightarrow{x} H_n(X \times Y) \xrightarrow{} \bigoplus_i \text{Tor}_R(H_i(X), H_{n-i-1}(Y)) \to 0$$

which splits (but not naturally).

Aside: Univar. char. $A \times B \xrightarrow{m} M \leftarrow R$-module $R$-bil. such that $\forall R$-bil. $A \times B \to C$ $\exists! R$-module hom $M \to C$

where

$$A \times B \xrightarrow{} M \xrightarrow{} C$$

commutes.