Typical object: Compact orient 3-manifold, either closed, such as $S^3$, $T^3 = S^1 \times S^1 \times S^1$, $L(p,q)$ (Surface) $\times S^1$, $UT(Surface)$, or with torus boundary, e.g. $S^3 \setminus N(Knot)$.

Geometric topology [study of manifolds]

High dim ($\geq 5$)
Well understood/impossible
\(\Rightarrow\) turned into homotopy theory

Low dim ($\leq 4$)

Dim 1: $\bigcirc$

Dim 2: $\bigcirc \begin{array}{c} \text{classification...} \\ \text{Dynamics/Moduli: } MCG(\Sigma) = \pi_0(\text{Diff}^+(\Sigma)) \\ \mathcal{J}(\Sigma) = \text{hyperbolic structures} \\ (\text{dim } 6g - 6) \end{array}$
Dim 4: $\pi_1$ arbitrary, crazy stuff happens:
uncountably many nondiffeomorphic smooth manifolds
that are homeo to $\mathbb{R}^4$. But: Closed $M^4$ with
$\pi_1 = 1$ are det. by cup product on $H^2(M; \mathbb{Z})$.
[Geometry rare.]

Dim 3: "Topology is geometry" (Thurston/Pennerman)
All closed $M^3$ have "geometric decompositions" with a
"generic" piece having a hyperbolic metric (const
curvature -1); such are unique (Mostow).
Cor: $\pi_1 M^3$ is residually finite: $\bigcap H = \{1\}$
$[\pi_1 M : H] < \infty$

[Interested? Stay for 595 AT 3!]

Foliations: $M^n$ smooth
$F$ partition of $M$ into subsets (leaves)
locally like $\mathbb{R}^k \times \{\text{pt}\}$ in $\mathbb{R}^n$. 
Examples: 1) $M = \mathbb{R}^n$, $\mathcal{F} = \bigsqcup_{y \in \mathbb{R}^{n-k}} \mathbb{R}^k \times \{y\}$

2) $M = A^k \times B^{n-k}$, $\mathcal{F} = \{A \times b \mid b \in B\}$

3) $M \xrightarrow{f} B$ a smooth submersion

$\mathcal{F} = \{f^{-1}(b) \mid b \in B\}$  [Includes (2)]

Ex: $f : \mathbb{R}^2 \to \mathbb{R}$

$f(x, y) = (x^2 - 1)e^y$

$f_x = 2xe^y$, $f_y = (x^2 - 1)e^y$

$f = 0: x = \pm 1$

$f = -1: e^y = \frac{1}{1-x^2}$

$f = 1: e^y = \frac{1}{x^2-1}$

Note

$f(x, y + s) = e^s f(x, y)$

So shifting

$f = e^s c$ up by $s$

takes it to $\{f = e^s c\}$  [Note: Not a product or fiber bundle.]
4) \( M = \mathbb{R}^2 / \mathbb{Z}^2 \), \( a \in \mathbb{R} \).
\( \mathbb{R}^2 \cong \tilde{F} = \{ \text{lines of slope } a \} \)
preserved by \( \mathbb{Z}^2 \to F \) on \( M \)
If \( a \notin \mathbb{Q} \), every leaf of \( \tilde{F} \) is dense.

5) A vector field \( X \) on \( M^n \) that is never 0 has
flow lines forming a 1-d foliation. \( (\Rightarrow X(M) = 0) \).

\[ \text{We'll focus on 2D fol of } M^3. \]

a) Every closed orientable \( M^3 \) has such a foliation.

b) \( \tilde{F} \) is taut when \( \exists \) a transverse closed loop meeting each leaf.

\[ \Rightarrow M = S^2 \times S^1, \mathbb{RP}^3 \# \mathbb{RP}^3, \text{ or } \tilde{M} \text{ min} = \mathbb{R}^3 \]
\( (\pi_1 \text{ is infinite}) \)
When co-orientable

\[ \Rightarrow \text{Every cpt leaf, } L \text{ is the simplest surface in its homology class.} \]
⇒ $M$ is a "contact bdry comp" of some symplectic $W$. 

⇒ $\hat{\text{F}}_{\text{red}}(M) \neq 0$

⇒ $\pi_1 M$ has a total order inv. under left mult.

Focus: a) Examples / Constructions

b) Broad picture / Big ideas

Not included: Many detailed proofs.

Go over syllabus, etc....