Lecture 17: More on the L-space Conjecture.

Conj: $M^3$ closed orient irreducible, TFAE

1) $M$ has a co-orient taut foliation
2) $\pi_1 M \cong \text{Homeo}^+(\mathbb{R}) \iff \pi_1 M$ is left-orderable
3) $M$ is not an L-space, i.e. $\hat{HF}_{\text{red}}(M) \neq 0$

Prop: $G$ is left-orderable $\iff G = P \amalg \mathbb{Z} \amalg P^{-1}$

$P \subseteq P$ with $P, P \subseteq P$

Pf ($\Rightarrow$): Set $P = \{ g \in G \mid g > 1 \}$. Note
$g > 1 \iff 1 > g^{-1}$ and $g, h > 1 \implies g \cdot h > g \cdot 1 > 1$.
$(\Leftarrow)$ Say $g > h \iff h^{-1} g \in P$.

Ex: Each line with irrational slope in $\mathbb{R}^2$ gives a left-order on $\mathbb{Z}^2$

Ex: $G = \pi_1 (\text{Weeks mfld})$

$= \langle a, b \mid a b a b a b^{-1} a^2 b^{-1}, b a b a b a^{-1} b^2 a^{-1} \rangle$

[Smallest volume hyp 3-mfld $0.94\ldots$]

is not left-orderable.

Pf: Can assume $a \in P$. If $b^{-1} \in P$, then $b^{-1} R_2 b R_1 = b^{-1} a b a b^{-1} a^2 b^{-1} \in P$, a contradiction since $1 \notin P$. 
Assume b ∈ P. [Hiding: a, b, ab⁻¹ ≠ 1]
  * If ab⁻¹ ∈ P, then so is abab(ab⁻¹)a(ab⁻¹).
  * If ba⁻¹ ∈ P, then so is baba(ba⁻¹)b(ba⁻¹).

In both cases, 1 ∈ P a contradiction.

[ A non-left orderable G which is finitely generated solvable word problem has a proof of non-orderability "like this". ]

Thm: Suppose $M^3$ is closed orient irreducible. If $\pi_1 M$ has a non-trivial homomorphism to $\text{Homeo}^+(\mathbb{R})$ then $\pi_1 M$ is left-orderable.

Pf: Cor of: every f.g. subgroup of a f.g. G has a hom. to $\text{Homeo}^+ G \Rightarrow G$ is left-orderable.

Cor: The Weeks manifold does not have a taut fol with $\hat{\mathbb{L}} \cong \mathbb{R}$.

Pf: Since $M^3 \cong (\mathbb{C} \times S^1)^2$, any fol is co-orient.

So get $\pi_1 M \to \text{Homeo}^+ (\mathbb{R})$ from action on leaf space. So $\pi_1 M$ is left-orderable by thm, a contradiction.
Basic tool:

\[ \rho: \pi_1 M \to \text{PSL}_2 \mathbb{R} \leq \text{Homeo}^+(S') \]

Define \( \widehat{\text{Homeo}^+(S')} \leq \text{Homeo}^+(\mathbb{R}) \) to be the subgroup of lifts of \( \text{Homeo}^+(S') \). Equiv, is those elts commuting with action of \( \mathbb{Z} \) by translation. Have \( \circ \)

\[ 0 \to \mathbb{Z} \to \widehat{\text{Homeo}^+(S')} \to \text{Homeo}^+(S') \to 1 \]

a central extension. Given \( \rho: \pi_1 M \to \text{Homeo}^+(S') \), can lift to \( \widehat{\text{Homeo}^+(S')} \) iff \( e(\rho) \in H^2(M; \mathbb{Z}) = 0 \) where \( e(\rho) \) is the Euler class (see Foll II, Ch 4).

Remark: \( \rho \) lifts to \( \widehat{\text{Homeo}^+(S')} \) \( \iff \) the (flat) circle bundle over \( M \) assoc. to \( \rho \) has a section. For the unit circle bundle of a \( C^* \) line bundle, \( e = C_1 \).

Cor: If \( H_1(M; \mathbb{Z}) = 0 \) and \( M \) has a taut fol, then \( \pi_1 M \) is left-orderable. \( Pf: \) Univ. Circle.
Then: The Weeks manifold does not have a taut fol.

Pf: Since $H_1(W; \mathbb{Z}) = (\mathbb{Z}/5)^2$, the Euler class of $\pi_1 W \to \text{Homeo}^+(S^1)$ has order 1 or 5 in $H^2(W; \mathbb{Z})$. Can't have $e(\pi_{univ}) = 0$ since $\pi_1 W$ is not left-orderable. Hence there exists an index 5 subgp $\Gamma \leq \pi_1 W$ so $e(\pi_{univ} \mid \Gamma) = 0$ and so $\Gamma$ is left-orderable. A similar but more involved arg. shows this is impossible. See [CD 2003].

Let $\widetilde{\text{PSL}_2 \mathbb{R}}$ be the preimage of $\text{PSL}_2 \mathbb{R}$ in $\text{Homeo}^+(S^1)$. It is also the universal covering Lie gp of $\text{PSL}_2 \mathbb{R} \cong \text{UT}(H^2) \cong \text{homeo}. S^1$. So

$$0 \to \mathbb{Z} \to \widetilde{\text{PSL}_2 \mathbb{R}} \to \text{PSL}_2 \mathbb{R} \to 1$$

Any hyp $M^3$ has $\pi_1 M \to \text{Isom}^+(H^3) \cong \text{PSL}_2 \mathbb{C}$.

By local rigidity, image conjugate into $\text{PSL}_2 \mathbb{K}$ for a number field $\mathbb{K}$. If $\mathbb{K}$ has a real embedding, get $\pi_1 M \leq \text{PSL}_2 \mathbb{R}$. 
For the 300k mfd sample, found an average of 7.9 $SL_2\mathbb{R}$ reps per mfd. Enough have $e = 0$ that some 64,000 have left-orderable $\pi_1$. All were non-L-spaces, const. w/ conjecture. If $e(p) = 0$ with prob $1/|H^2(M; \mathbb{Z})|$ except 6,300 counterexamples ($p \approx 10^{-2.700}$)

[Boyer-Hu] $e(p_{univ}) = e(T\Sigma)$ for a co-orientable taut fol.

Can use to show > 32,000 of these mfdls have orderable $\pi_1$; more than 160,000 have taut fol.