Q: What is simplest rep of a homology class?

Setting: $M^3$ clsd orient irreducible.

Any class in $H_2(M; \mathbb{Z})$ can be rep by a smooth clsd surface since $H_2(M) \cong H'_1(M) = [M, S']$ so can take $S = f^{-1}(pt)$ for $f: M \to S'$. An oriented surface $S \subseteq M$ is nice when no component is $0$ in $H_2(M)$. In particular, every comp of $S$ has $\chi \leq 0$. Define

$$\| c \|_{Th} = \min \left\{ -\chi(S) \mid S \text{ is a nice rep of } c \right\}$$

Then a) $\| a + b \|_{Th} \leq \| a \|_{Th} + \| b \|_{Th}$

b) $\| k \cdot c \|_{Th} = |k| \cdot \| c \|_{Th}$

c) When $M$ is orientable, $\| c \|_{Th} = 0 \iff c = 0$.

A nice surface is taut when $|\chi(S)| = \| [S] \|_{Th}$. 
Lemma: A taut surface $S$ is incompressible.

Pf: Assume $S$ is connected. [Don't really use, just makes notation simpler.] Suppose $D$ is a compressing disc for $S$. Let $S' = (S \setminus N(\partial D)) \cup D_1 \cup D_2$

$\partial S' = \partial D$

which has $[S'] = [S]$ and $\chi(S') = \chi(S) + 2$

If $\partial D$ does not sep $S$, then $S'$ is connected, so $S'$ is nice as $S$ is, but this contradicts that $S$ is taut. So $\partial D$ does sep $S$ and $S' = S_1 \cup S_2$, which can't be nice. So say $[S_1] = 0$. If $\chi(S_1) \leq 0$ then $[S_2] = [S]$ with $-\chi(S_2) \leq -\chi(S) - 2$ again violating that $S$ is taut. So $S_1 = \emptyset$

$\Rightarrow \partial D$ bounds a disc in $S$. So $S$ is incomp.

Pf of $\emptyset$: Immediate from lemma.
Pf of (a): Let $A, B$ be taut surf for $a, b$.

As they are incomp, can isotope so are transv and every component of $A \cap B$ is essential in both $A$ and $B$.

Idea: If $C \subseteq A \cap B$ bounds disc in $A$ it must also bound one in $B$. These two discs bound a ball in $M$, and isotope across this to reduce $\#A \cap B$.

So every comp. of $A \setminus B$ and $B \setminus A$ has $\chi \leq 0$.

Let $C$ be the orient sum of $A$ and $B$

No comp of $C$ is $S^2$,

so if $C'$ is the nice surface obtained by deleting any sep. comps, we have

$$-\chi(C') \leq -\chi(C) = -\chi(A) - \chi(B)$$

Since $[C'] = a + b$, get $\|a + b\|_{Th} \leq \|a\|_{Th} + \|b\|_{Th}$.

Pf of (b): Hint: Show any $S$ rep $K$ a

consists of $K$ surfaces each rep $a$. 

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Thm: \( \| \cdot \|_{\mathcal{H}} \) extends from \( H_2(M;\mathbb{Z}) \) to a norm on \( H_2(M;\mathbb{R}) \). Its unit ball is a finite rational polytope.

Idea: Extend to \( H_2(M;\mathbb{Q}) \) by making it linear on rays. 

Then extend by cont to \( H_2(M;\mathbb{R}) \). Only uses that \( \| \cdot \|_{\mathcal{H}} \) takes integer values on \( H_2(M;\mathbb{Z}) \).

Thm: Suppose \( S \) is a compact leaf of a co-orient taut fol \( \mathcal{F} \). Then \( S \) is taut.

[Gabai] Suppose \( S \) is a taut surface in a closed orient irreducible \( M^3 \). Then \( \exists \) a co-orient taut fol \( \mathcal{F} \) with \( S \) as a compact leaf.

Cor: If define \( \| \cdot \|_{\mathcal{H}} \) using immersed surfaces or just \( \mathcal{F} \rightarrow M \) get the same norm.
Thm: Suppose $F$ is a cpt leaf of a co-orient taut $\mathcal{F}$. Then $F$ is taut.

Pf idea: Suppose $S$ is any taut surface with $[S] = [F]$. As $S$ is incomp, can homotope $S$ so it is trans to $\mathcal{F}$ except at finitely many saddle tangencies. Two kinds, dep on whether $T_p S$ and $T_p F$ have the same orient. Set $I_p = \#$ where agree and $I_n = \#$ where disagree. Recall $I_p + I_n = -\chi(S)$.

Lemma: If $e(T\mathcal{F})$ is the Euler class of $T\mathcal{F}$ in $H^2(M)$ then $e(T\mathcal{F})([S]) = I_n - I_p$

Assuming this,

$$\| [S] \|_{T_n} = -\chi(S) = I_p + I_n \geq I_p - I_n$$

$$= -e(T\mathcal{F})([S]) = -e(T\mathcal{F})([F])$$

$$= -e(T\mathcal{F}|_{\mathcal{F} = TF})([F])$$

$$= -\chi(F).$$

So $F$ is also taut. (no cusp of $F$ is sep since $\mathcal{F}$ has a c1sc(t) trans.)