Lecture 19: Foliations and the Thurston Norm

[Goal for rest of class: constructing taut fol.]

Q: What is simplest rep of a homology class?

Setting: M^3 clsd orient irreducible.

Any class in $H_2(M;\mathbb{Z})$ can be rep by a smooth clsd surface since $H_2(M) \cong H^1(M) = [M, S^1]$ so can take $S = f^{-1}(pt)$ for $f: M \to S^1$. An oriented surface $S \subseteq M$ is nice when no component is 0 in $H_2(M)$. In particular, every comp of S has $\chi \leq 0$. Define

$$\|c\|_{Th} = \min\{-\chi(S) \mid S \text{ is a nice rep of } c\}$$
$$-\chi(S) = |\chi(S)|$$

Then a) $\|a+b\|_{Th} \leq \|a\|_{Th} + \|b\|_{Th}$

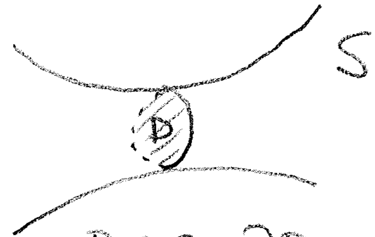
b) $\|kc\|_{Th} = |k| \, \|c\|_{Th}$

c) When M is atoroidal, $\|c\|_{Th} = 0 \iff c = 0$.

A nice surface is taut when $|\chi(S)| = \|[S]\|_{Th}$.

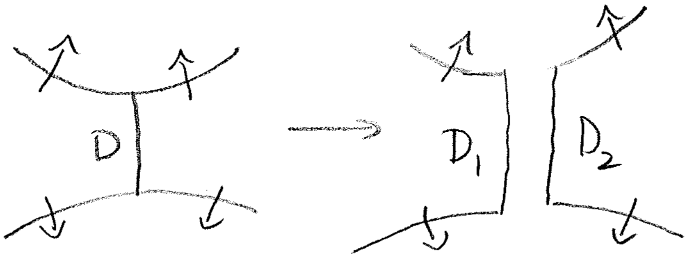
Lemma: A taut surface S is incompressible.

Pf: Assume S is connected. [Don't really use, just makes notation simpler.] Suppose D is a compressing disc for S. Let



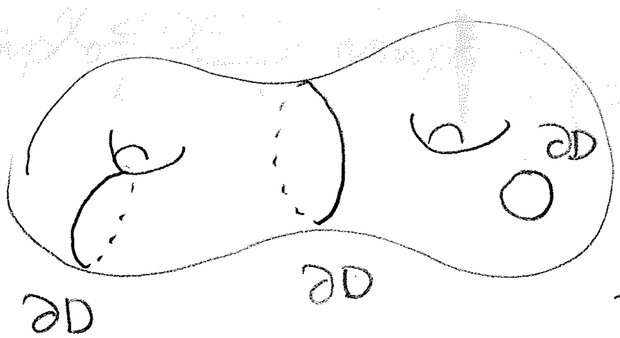
$D \cap S = \partial D$

$$S' = (S \setminus N(\partial D)) \cup D_1 \cup D_2$$



which has $[S'] = [S]$ and $\chi(S') = \chi(S) + 2$

If ∂D does not sep S, then S' is connected, so S' is nice as S is; but this contradicts that S is taut.



So ∂D does sep S and $S' = S_1 \cup S_2$, which can't be nice. So say $[S_1] = 0$. If $\chi(S_1) \leq 0$ then $[S_2] = [S]$ with $-\chi(S_2) \leq -\chi(S) - 2$ again violating that S is taut. So $S_1 =$ 

$\Rightarrow \partial D$ bounds a disc in S. So S is incomp. $\blacksquare$

Pf of ©: Immediate from lemma. $\blacksquare$

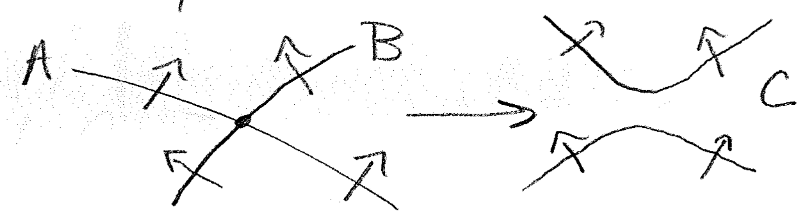
Pf of a : Let A, B be taut surf for a, b.

As they are incomp, can isotope so are transv and every component of $A \cap B$ is essential in both A and B.

[Idea: If $C \subseteq A \cap B$ bounds disc in A it must also bound one in B. These two discs bound a ball in M, and isotope across this to reduce $\# A \cap B$.]

So every comp. of $A \setminus B$ and $B \setminus A$ has $\chi \leq 0$.

Let C be the orient sum of A and B



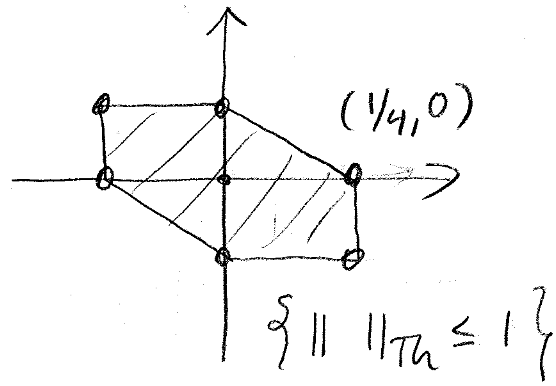
No comp of C is S^2, so if C' is the nice surface obtained by deleting any sep. comps, we have

$$-\chi(C') \leq -\chi(C) = -\chi(A) - \chi(B)$$

Since $[C'] = a + b$, get $\|a+b\|_{Th} \leq \|a\|_{Th} + \|b\|_{Th}$ ▨

Pf of b : Hint: Show any S rep $k \cdot a$ consists of k surfaces each rep a.

Thm: $\| \cdot \|_{Th}$ extends from $H_2(M;\mathbb{Z})$ to a norm on $H_2(M;\mathbb{R})$. Its unit ball is a finite rat'l polytope.



Idea: Extend to $H_2(M;\mathbb{Q})$ by making it linear on rays.

Then extend by cont to $H_2(M;\mathbb{R})$. Only uses that $\| \ \|_{Th}$ takes integer values on $H_2(M;\mathbb{Z})$. ▨

Thm: Suppose S is a compact leaf of a co-orient taut fol $\mathcal{F}$. Then S is taut.

[Gabai] Suppose S is a taut surface in a clsd orient irred ator. M^3. Then $\exists$ a co-orient taut fol $\mathcal{F}$ with S as a compact leaf.

Cor: If define $\| \ \|_{Th}$ using immersed surfaces or just $F \to M$ get the same norm.

Thm: Suppose F is a cpt leaf of a co-orient taut $\mathcal{F}$. Then F is taut.

Can allow immersed!

Pf idea: Suppose S is any taut surface with $[S]=[F]$. As S is incomp, can homotope S so it is trans to $\mathcal{F}$ except at finitely many saddle tangencies. Two kinds, dep on whether T_pS and $T_p\mathcal{F}$ have the same orient. Set $I_p = \#$ where agree and $I_n = \#$ where disagree. Recall $I_p + I_n = -\chi(S)$.

Lemma: If $e(T\mathcal{F})$ is the Euler class of $T\mathcal{F}$ in $H^2(M)$ then $e(T\mathcal{F})([S]) = I_n - I_p$

Assuming this,

$$\begin{aligned}\|[S]\|_{Th} &= -\chi(S) = I_p + I_n \geq I_p - I_n \\ &= -e(T\mathcal{F})([S]) = -e(T\mathcal{F})([F]) \\ &= -e(T\mathcal{F}|_F = TF)([F]) \\ &= -\chi(F).\end{aligned}$$

So F is also taut. (no comp of F is sep since $\mathcal{F}$ has a clsd trans.) ▨