Lecture 23: Thurston norm and foliations.

Last time: $M^3$ closed orient irreducible. For $c \in H_2(M; \mathbb{Z})$ set
\[
\|c\|_{Th} = \min \left\{ -\chi(S) \mid S \text{ a nice surface (no comp is 0 in } H_2) \right\}
\]
that reps $c$.

A nice $S$ is taut when $-\chi(S) = \|S\|_{Th}$.

Thm: Suppose $F$ is a cpt leaf of a co-orient taut $\mathcal{F}$.
Then $F$ is taut.

Can allow immersed!

Proof idea: Suppose $S$ is a taut surface rep $[F]$.
As $S$ is incomp, can homotope $S$ so transv to $\mathcal{F}$ except at finitely many saddle tangs. Two kinds depend on whether $T_p S$ and $T_p F$ have the same orient. Set $I_p = \#$ orients agree, $I_n = \#$ disagree.

Recall $I_p + I_n = -\chi(S)$.

Lemma: If $e(TF) \in H^2(M)$ is the Euler class of $TF$ then $e(TF)([S]) = e(TF|_S)([S]) = I_n - I_p$.

Proof: See either text but in a picture:
$F$ co-orient
$\Rightarrow F \cap S$ co-orient
$\Rightarrow F \cap S$ orient. $\Rightarrow$ gives section of $TF|_S$ van. at tang.
Now
\[ \| [s] \|_{T_n} = -\chi(s) = I_p + I_n \geq I_p - I_n \]
\[ = -e(T_F)([s]) = -e(T_F)([F]) \]
\[ = -e(T_F|_F = TF)([F]) = -\chi(F) \]

No comp of \( F \) sep, since each is meet by an orient cld trns. So \( F \) is also taut.

Cor: For \( M_f \), each \( F \times \mathbb{R}^3 \) is taut.

Thm: \( M \) atoroidal so \( B_T = \{ c \in H_2(M; \mathbb{R}) \mid \| c \|_{T_n} \leq 1 \} \)
is a finite rat l polytope. There exist top dim faces \( A_i \) of \( B_T \) such that

1. \( c \in H_2(M; \mathbb{Z}) \) can be rep by a fiber in a fib over \( S^1 \)
   \[ \iff c/\|c\|_{T_n} \text{ in some } \text{int}(A_i). \]

2. \( \phi \in H^1(M; \mathbb{R}) \) can be rep by a nowhere vanishing 1-form
   \[ \iff c = \text{PD}(\phi) \text{ has } c/\|c\|_{T_n} \text{ in some } \text{int}(A_i). \]

Cor: If \( b_1 > 1 \), then some \( c \neq 0 \) in \( H_2(M; \mathbb{Z}) \) cannot be rep by a fiber.
Notes: 1) If \( p: M \rightarrow S' \) is a fib with fiber \( F \), then \( \text{PD}(F) \) is rep by \( p^*(d\Theta) \) which is never 0.

2) The set of \( \emptyset \) rep by nowhere 0 forms is open.

3) If \( \phi \in H^1(M; \mathbb{Z}) \) rep by nowhere 0 \( \omega \), then int. \( \omega \) gives a fib \( p: M \rightarrow S' \).

[Gabai] If \( F \) is taut, \( \exists \) a co-orient \( \widetilde{F} \) with \( F \) as a leaf.

Cor: Suppose \( F \subseteq M \) is taut. If \( \widetilde{M} \rightarrow M \) is a finite cover and \( \widetilde{F} \) a component of \( p^{-1}(F) \), then \( \widetilde{F} \) is taut.

Pf: Let \( F \) be taut with \( F \) as a leaf. Then \( p^{-1}(F) \) is taut (why?) and has \( \widetilde{F} \) as a leaf.
Won't be able to say anything about the proof of Gabai's lemma in time remaining...

Story time: $M$ is closed, irreducible. If $H_2(M; \mathbb{Z}) \neq 0$, then $|\pi_1 M| = \infty$ since $C \neq 0$ can be represented by an incompressible surface where every component has genus at least 2. So $M$ is hyperbolic, the metric is unique. [What about geometric complexity of $C$?]

Mot by $\langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta$, on $H_1(M; \mathbb{R})$ set

$$\| \phi \|_{L^2}^2 = \inf \left\{ \int_M | \alpha |^2 \mid \alpha \text{ reps } \phi \right\} = \langle \omega, \omega \rangle$$

where $\Delta \omega = 0$.

By P.D., have $\| \cdot \|_{Th}$ on $H_1(M; \mathbb{R})$ as well.

[Bergeron-Sengün-Venkatesh; Brock-D 2017]

$$\frac{\pi}{\sqrt{\Vol(M)}} \| \phi \|_{Th} \leq \| \phi \|_{L^2} \leq \frac{10\pi}{\sqrt{\inj(M)}} \| \phi \|_{Th}$$

where $\inj(M) = \frac{1}{2} \min(\text{len}(\text{closed geod}))$.

[If time remains, do least area norm.]