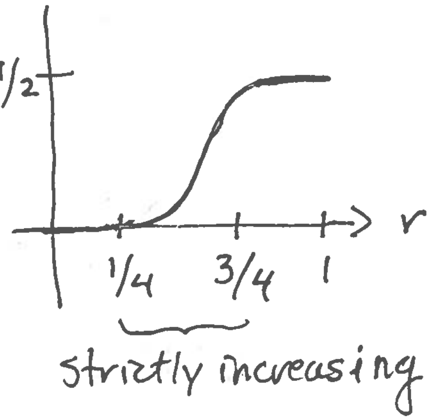
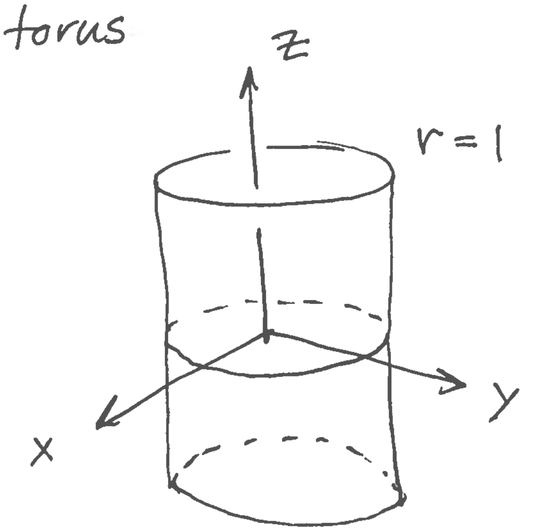
Lecture 5: Reeb stability

Last time: Improved Reeb solid torus

λ smooth on $[0,1]$





In cylindrical coor, set

$$\omega = -\sin(\lambda(r))\,dr + \cos(\lambda(r))\,dz$$

(Note $\omega \wedge d\omega = 0$ and θ-indep.)

In (x,z) plane with $x \geq 0$, have

$$\cos(\lambda(r))\frac{\partial}{\partial x} + \sin(\lambda(r))\,\partial/\partial z$$

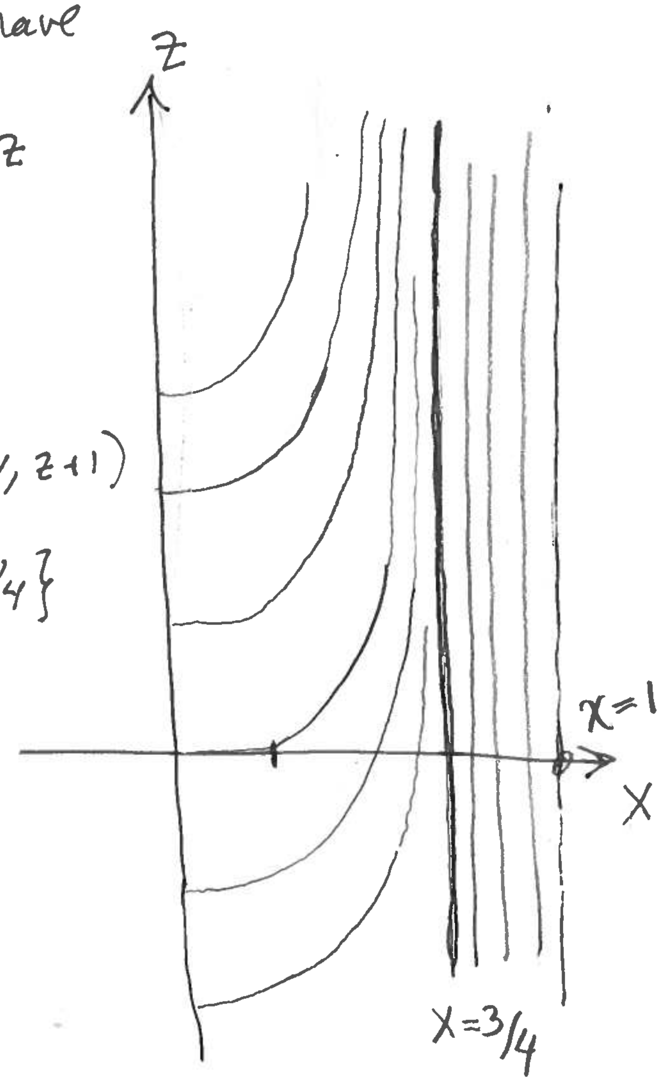
is tangent to the leaves.

Set $M_0 = \{r \leq 1\} \big/ (x,y,z) \mapsto (x,y,z+1)$

and $M_1 \subseteq M_0$ where $\{r \leq 3/4\}$

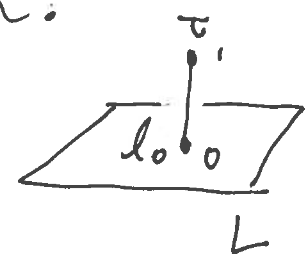
Then M_1 has a "Reeb fol"

[∂M_1 a leaf, others spiral out to it.]



and $\overline{M_0 \setminus M_1}$ is $[3/4, 1] \times T^2$ fol by $\{pt\} \times T^2$.

Def: $\mathcal{F}$ has infinitesimally trivial holonomy along a tangential component L of ∂M when:
For a trans. τ at $l_0 \in L$ param by $[0,1]$ and all $\alpha \in \pi_1(L, l_0)$ if $h = hol_\alpha$ is a diff $[0,a] \to [0,b]$ then $h'(0) = 1$ and $h^{(k)}(0) = 0$ forall $k \geq 2$.



Ex: Actual triv. holonomy (each $hol_\alpha = id_{[0,c]}$), e.g. a product fol $\Sigma \times S^1$.

Non Ex: Original Reeb comp: $hol_\alpha : t \mapsto 1/2\, t$

Ex: Improved Reeb comp since can compute hol along $r = 3/4$ torus from the outside where hol is the identity.

Have the following gluing thm that gives us the promised smooth fol on S^3:

[Prop 3.4.2 of Fol I] Suppose N_i is fol by $\mathcal{F}_i$ with S_i a tang. comp of ∂N_i.
If both $\mathcal{F}_i$ have inf. triv. hol along S_i and $\varphi: S_1 \longrightarrow S_2$ is a diffeom, then
$N = N_1 \cup_\varphi N_2$ is smoothly fol by $\mathcal{F}_1 \cup \mathcal{F}_2$.

[Soon will foliate all clsd M^3 using similar techniques.]

<u>Reeb Stability</u>: Suppose a compact leaf L of $\mathcal{F}$ on M^3 has trivial holonomy [hol around all csld loops = id]
Then L has an open nbhd $N \cong L \times (-1,1)$ where $\mathcal{F}$ is the product fol.

<u>Pf idea</u>: L is compact, so cover w/ finitely many fol. charts and use triv. hol to build N.
[See Thm 2.4.1 in Fol I for details] ☑

<u>Reeb Stability</u>: Suppose $\mathcal{F}$ is a co-orient fol of a clsd conn. orient. M^3. If some leaf of $\mathcal{F}$ is S^2, then $M = S^2 \times S^1$ with the product fol.

Note: If $\mathcal{F}$ is not co-orient, there is one more poss: $\mathbb{R}P^3 \# \mathbb{R}P^3$. If M^3 is not orient, add $S^2 \tilde{\times} S^1$ = Mapping Torus of anitpodal map (see last pg).

Claim: $\mathbb{R}P^3 \# \mathbb{R}P^3 = S^2 \times S^1 / (v,z) \mapsto (-v, \bar{z})$ $\quad S^2 \subseteq \mathbb{R}^3$, $S^1 \subseteq \mathbb{C}$

where the product fol descends.

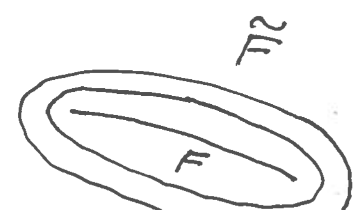
Construction: Suppose F is a connected nonorient surface, $\tilde{F} \to F$ the orient. cover, and $\tau : \tilde{F} \to \tilde{F}$ the assoc. involution. [e.g. $F = \mathbb{R}P^2$, $\tilde{F} = S^2$, $\tau = v \mapsto -v$]

Define $N(F) = \tilde{F} \times [-1,1] / (x,t) \mapsto (\tau(x), -t)$

which has a fol by the images of $\tilde{F} \times \{pt\}$.

One leaf is a copy of F (image of $\tilde{F} \times \{0\}$)

all others, including $\partial N(F)$, are $\tilde{F}$.



[Compare: Replace $\tilde{F}, F$ by S^1 and make $\tilde{F} \to F$. Then $N(F)$ is the Möbius
$z \mapsto z^2$ band covered by the annulus $S^1 \times [-1,1]$.]

Exercise: $N(F)$ is the unique I-bundle over F with orient. total space. Any $F \subseteq$ orient M^3 has a nbhd $\cong N(F)$.

Now $\mathbb{R}P^3 = S^3 / {v \mapsto -v}$ and $\mathbb{R}P^3 \setminus$ open ball has preimage $S^3 \setminus$ (open balls about north & south poles) $\cong S^2 \times [-1,1]$. Restricting the antipodal map shows $\mathbb{R}P^3 \setminus$ open ball $\cong N(\mathbb{R}P^3)$.

The fol. of $S^2 \times S^1 / (v,z) \mapsto (-v, \bar{z})$ has two $\mathbb{R}P^2$ leaves sep by S^2 leaves, i.e. is $N(\mathbb{R}P^2) \cup N(\mathbb{R}P^2)$, proving the claim.

Construction: Suppose F is a surface, $f: F \to F$ a diffeo. The mapping torus of f is

$$M_f = F \times [0,1] / (x,1) \to (f(x), 0)$$

$\boxed{\text{Fol by } F \times \{pt\} \text{ with trivial holonomy}}$

and say that such a 3-manifold fibers over S^1.