Lecture 12: Properties of taut foliations.

Last time: $\mathcal{F}$ of M^3 is taut when every leaf meets a clsd transversal.

Thm: Suppose $\mathcal{F}$ is co-orient fol of a clsd orient M^3. Then $\mathcal{F}$ is not taut $\iff$ $\exists$ torus leaves $T_1, \dots, T_k$ bounding a submfld V where the co-orient pts into V at each T_i.

Thm: $\mathcal{F}$ co-orient of clsd orient M^3. Then $\mathcal{F}$ is taut $\iff$ $\exists$ a flow trans. to $\mathcal{F}$ that pres. a vol form.

Note: A "dead end" V obstructs having such a flow.

[Novikov-Rosenberg] Let $\mathcal{F}$ be a taut fol of a clsd orient M^3. If $M \neq S^2 \times S^1$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$, then

1) M is irreducible
2) every leaf L is incomp ($\pi_1 L \hookrightarrow \pi_1 M$)
3) every clsd transversal is $\neq 1$ in $\pi_1 M$.

<u>Cor</u>: If M has a taut fol, then $\pi_1 M$ is infinite.
Let $\tilde{\mathcal{F}}$ be the fol of the univ. cover $\tilde{M}$ of M.
If $\tilde{M} \neq S^2 \times \mathbb{R}$, then every leaf of $\tilde{\mathcal{F}}$ is a properly emb. plane. [Same as $M \neq S^2 \times S^1$, $\mathbb{RP}^3 \# \mathbb{RP}^3$.]

<u>Pf of Cor</u>: Let γ be a closed transv. to $\mathcal{F}$.
Then γ^n can be perturbed to a clsd transv
$\Rightarrow \gamma^n \neq 1$ in $\pi_1 M \Rightarrow |\pi_1 M| = \infty$. By (2)
each leaf of $\tilde{\mathcal{F}}$ is a plane. If $\tilde{L} \hookrightarrow \tilde{M}$ is not proper, $\exists$ a fol chart in $\tilde{M}$ that it meets in at least 2 plaques $\Rightarrow \exists$ a clsd trans $\tilde{\gamma}$ to $\tilde{L}$.
The image γ of $\tilde{\gamma}$ in M is a clsd trans, so γ has infinite order in $\pi_1 M$. But then γ can't lift to $\tilde{\gamma}$. ▨

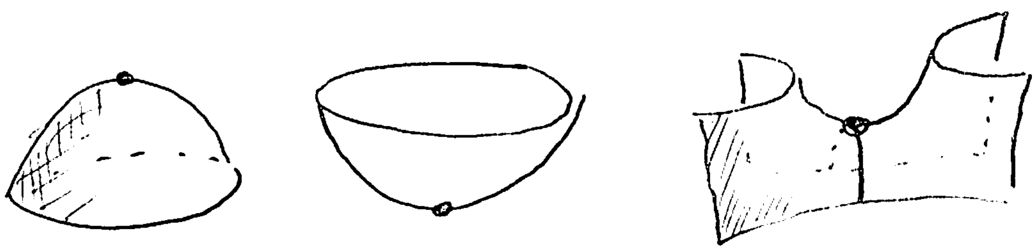
<u>Cor</u>: S^3, $L(p,q)$ have no taut foliations.

<u>Thm</u>: If a clsd orient M^3 contains no incomp. tori, a foliation $\mathcal{F}$ is taut $\Leftrightarrow$ it has no Reeb. comp.
<u>Cor</u>: Every fol of S^3 has a Reeb comp.

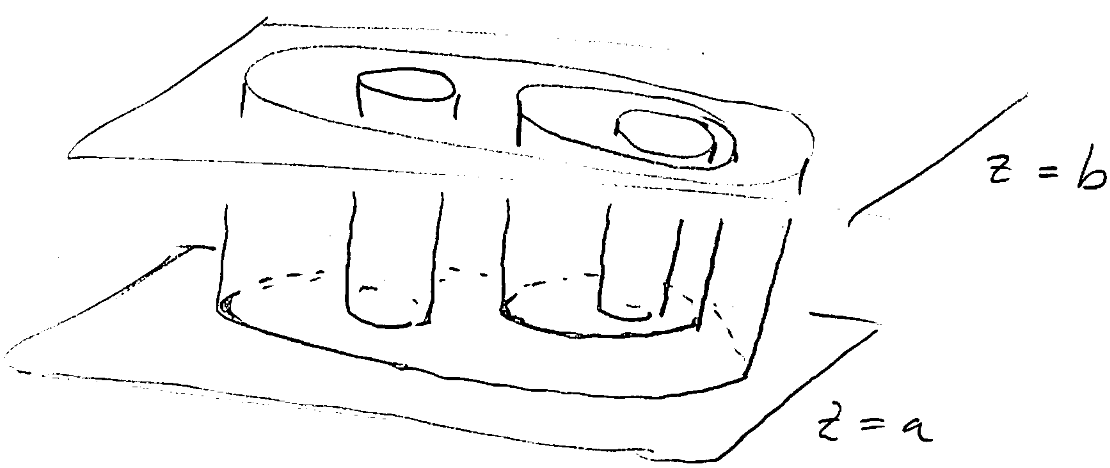
[Won't give full proof of N-R's thm, but will sketch some ideas...]

[Alexander] Every smooth S^2 in $\mathbb{R}^3$ bounds a ball, i.e. $\mathbb{R}^3$ is irred.

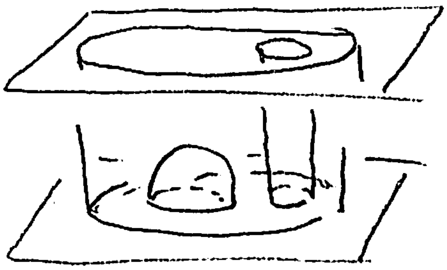
<u>Idea</u>: By perturb, can assume the z-coor restricts to a Morse fn on $F \cong S^2 \Rightarrow$ finitely many pts where $T_p F$ is horizontal, each of which is one of



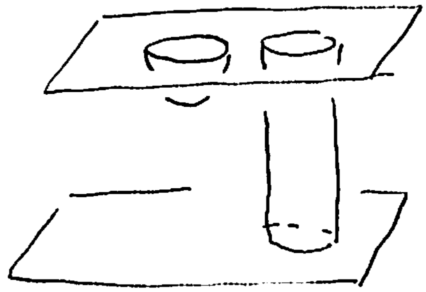
Can assume these are all at diff heights. Set $F^{[a,b]} = \{(x,y,z) \in F \mid z \in [a,b]\}$. If no crit pt with $z \in [a,b]$, then $F^{[a,b]}$ consists of "vertical annuli".



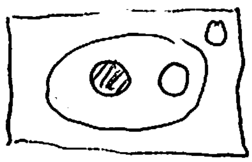
When $\exists!$ crit pt with $z \in [a,b]$ have one of:

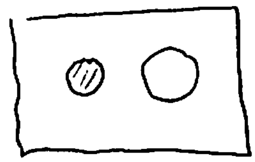


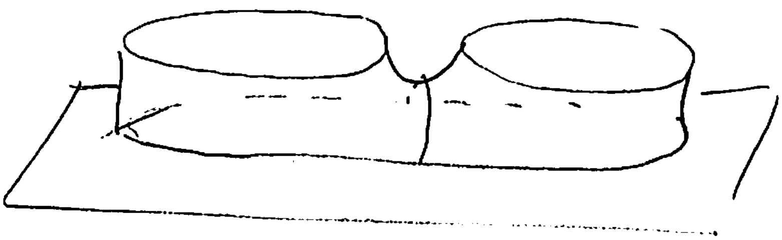
death

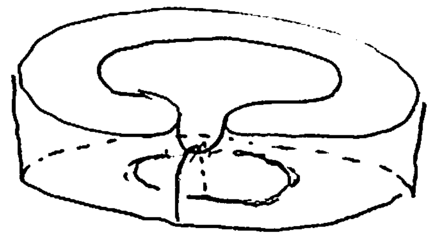


birth

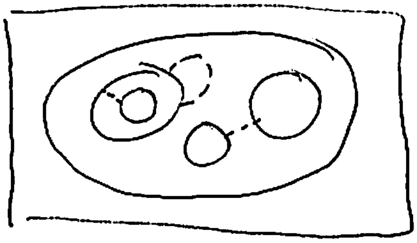








saddles

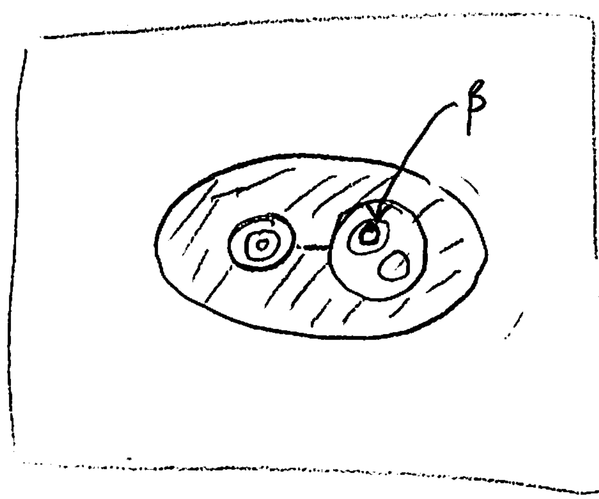


No saddle
$\Rightarrow$ one min, one max
$\Rightarrow$ std. S^2

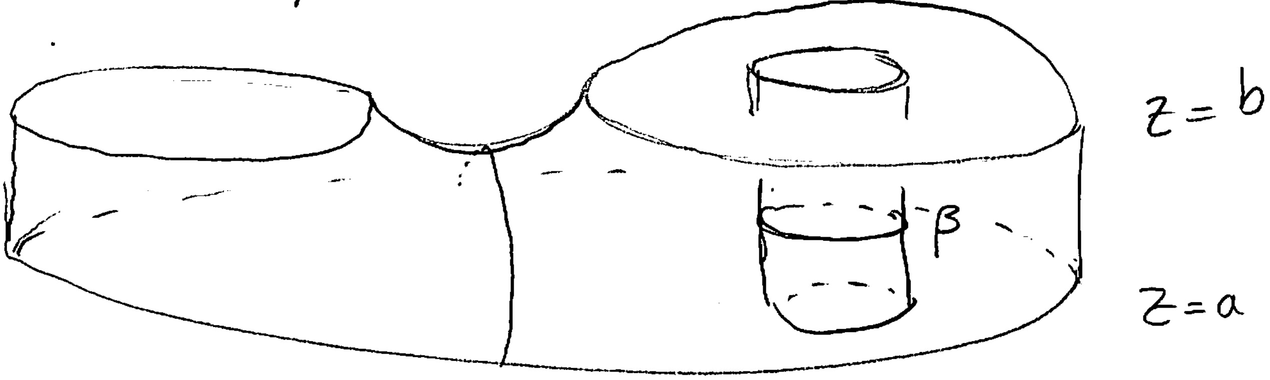
Iduct on # of saddles.

Look at some saddle

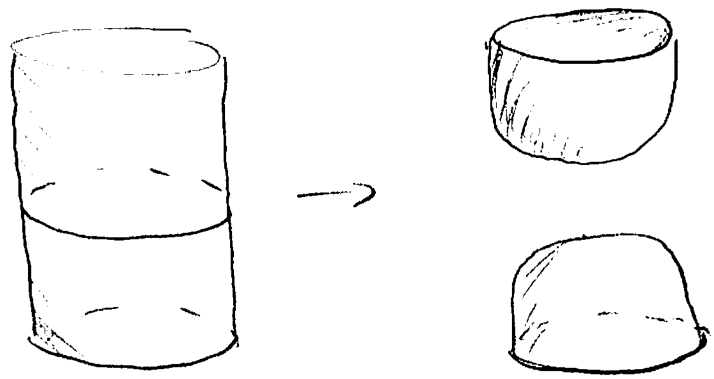
Take innermost curve inside innermost ∂ comp of the saddle



Picture when $\beta \not\subset \partial$saddle



Now surger along β:



This creates two surfaces F', F'' both spheres.

If one of F', F'' has no saddles, isotope F to reduce the # $\partial F^{[a,b]}$. Otherwise F' and F'' both have fewer saddles, bound balls in $\mathbb{R}^3$ by induct; use to show F also bounds.

If $\beta \subset \partial$sadde then F' or F'' having no saddles gives a local cancellation: