Lecture 12: More on taut foliations.

Thm Suppose $\mathcal{F}$ is a taut fol of a closed oriented $M^3$.
If $\mathcal{F}$ is not the std fol on $S^2 \times S^1$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$, then:

1) $M$ is irreducible.

2) Every leaf is incomp.

3) Every closed trans. is $\neq 1$ in $\pi_1 M$.

4) $\tilde{M}$ is $\mathbb{R}^3$ with $\mathcal{F}$ the product $\mathbb{S}^1 \times \mathbb{R}$
where $\mathbb{S}^1$ is a fol of $\mathbb{R}^2$ by lines. [Palmeira]

[Discuss drift approaches, references...]

Euler char of surfaces: Generic zeros of a vector field on $F$.

- Center
- Saddle

(Also minus these)

If $X$ is generic on $F$ and either trans or tang'snt to each comp of $\partial F$, then

$$\chi(F) = \sum_{\text{crit pt } p} I_p$$

$$(\text{of } X)$$

$I_p = \begin{cases} +1 & \text{center} \\ -1 & \text{saddle} \end{cases}$
Suppose $\mathcal{F}$ is a singular fol of $F$ with allowed sing.

\[ \chi(F) = \sum_{\text{sing } \rho} I_p. \]

Sketch of reason: By Sullivan, $\exists$ a metric on $M$ where every leaf is a minimal surface (mean curve $= 0$). If $M$ contains an ess. sphere, then $\exists$ an ess sphere of $F$ of least area (which is embedded) [Sacks-Uhlenbeck, Macks-Yau]. Assume $F$ is generic w.r.t. $\mathcal{F}$. By the barrier princ, there will be no center sing of $F \cap \mathcal{F}$.

Thus $\chi(F) \leq 0$ a contradiction.

Main non-generic case is $F$ is a leaf of $\mathcal{F} \Rightarrow M$ covered by $S^2 \times S^1$. 
Motivation: Reeb fol of $S^3$

The torus leaf is compact with

$$\mathcal{F} \cap D = \bigcirc$$

Each circle in $\mathcal{B}$ bounds a disc in a leaf of $\mathcal{F}$.

The areas of these discs go to $\infty$. They accumulate along the torus leaf, the only one that does not meet a closed trans.

The core is a closed transverse loop in $\pi_1 M$.

It bounds a disc $D'$ which meets $\mathcal{F}$ like

Idea in 2 and 3 is to study discs encoding that some $\gamma \subseteq L$ or trans is $1$ in $\pi_1 M$.

by the fol $D \Theta \mathcal{F}$. Since $\chi(D) = 1$ must be some center sing.
Near the center, each loop of $\mathcal{FN}D$ bounds a disc in a leaf $L$. This is an open cond, let $U \subseteq D$ be all loops of $\mathcal{FN}D$ with this prop.

Various poss:

a) $U = D \Rightarrow \partial D$ bounds a disc in its $L$ so not a real compression.

b) One component $l$ of $\partial U$ is a circle.
   $\Rightarrow$ "exploding disc" and a Reeb comp.

c) Something like

Now "cancel" this pair of sing.
We're only req. that these discs be immersed, so don't have to worry that there is some other part of it in the "toe of the clown shoe."

d) Variants of c)

Notes: Conclusions 2) and 3) only req $F$ is Reebless, not taut. Novikov also showed a Reebless fol $\Rightarrow \pi_2 = 0$ [weaker than irreducible, the Poincaré conj].

[Roussarie-Thurston] Suppose $F$ is taut and $S$ an immersed incompressible surface. Then $S$ is homotopic to either

a) a leaf of $F$.

b) intersect $F$ only in saddle tangencies