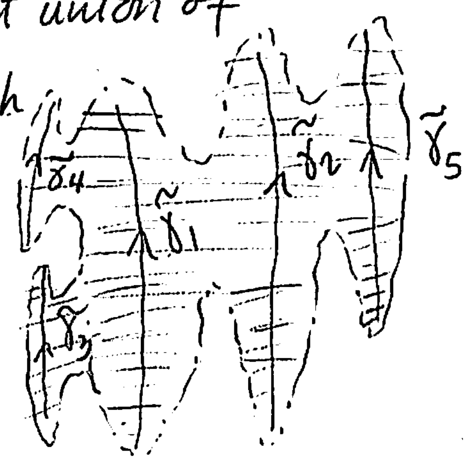
Lecture 14: Thurston's Universal Circle

Last time: If $\mathcal{F}$ is a taut fol of a clsd M^3, then the leaf space $\widetilde{\mathcal{L}}$ of $\widetilde{\mathcal{F}}$ in $\widetilde{M}$ is a simply connected "1-manifold".

poss non-Hausdorff.

In particular, $\pi_1 M$ acts on some s.c. "1-mfld" without a global fixed pt.

Pf: Let $\gamma \subseteq M$ be a clsd trans to $\mathcal{F}$ meeting every leaf. Then $p^{-1}(\gamma) \subseteq \widetilde{M}$ is a disjoint union of prop. embed. open transversals $\tilde{\gamma}_i$ which collectively meet every leaf of $\widetilde{\mathcal{F}}$.



Each $\tilde{\gamma}_i \hookrightarrow \widetilde{\mathcal{L}}$ and some conj of $[\gamma]$ in $\pi_1 M$ trans. along $\tilde{\gamma}_i$
$\Rightarrow$ no global fixed pt. ▨

Remark: Typically $\pi_1 M \longrightarrow \mathrm{Homeo}(\widetilde{\mathcal{L}})$ is injective.

Exception: M_f for $f : F \circlearrowleft \cong$.

Then $\widetilde{M}_f = \widetilde{F} \times \mathbb{R}$ and $\widetilde{\mathcal{L}} = \mathbb{R}$, with

$\pi_1 M \longrightarrow (\text{transl. by } \mathbb{Z}) \subseteq \mathrm{Homeo}(\mathbb{R})$

corr. to $\phi \in H^1(M; \mathbb{Z})$ that is Poincaré dual to $F \times \{pt\}$.

<u>Note</u>: $\exists$ fol with $\tilde{\mathcal{L}} = \mathbb{R}$ where the action is faithful, e.g. T^3 fol by planes of irrat. slope.

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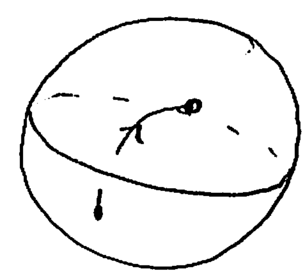
atoroidal: $\pi_1 M$ does not contain $\mathbb{Z}^2$. ($\Rightarrow$ no incomp tori.)

<u>Geometrization</u>: M^3 clsd orient irred and atoroidal. If $\pi_1 M$ is infinite then M is hyperbolic.

<u>Note</u>: If M is hyp, then $\tilde{M} = \mathbb{H}^3 \cong \mathbb{R}^3$ is irred $\Rightarrow M$ is irred. Also $\pi_1 M \leq \mathrm{Isom}^+(\mathbb{H}^3) \cong PSL_2\mathbb{C}$. $P^1(\mathbb{C})$ consists only of hyp. elts and any discrete subgp of such is cyclic.



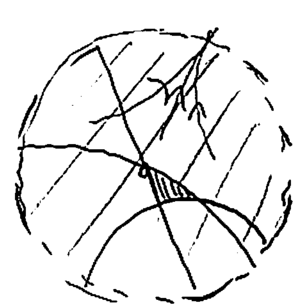
<u>Ex</u>: 1) All but 10 Dehn surgeries on . $\mathbb{H}^3 = \{|\vec{x}| < 1\}$

2) F^2 with $g \geq 2$, $\underbrace{f \text{ a pseudo-Anosov}}$, then M_f is hyp.

$\hookrightarrow$ e.g. f_* on $H_1(F^2; \mathbb{Z})$ has irred. char poly, not cyclotomic, all coeff nonzero.

<u>Univ. Circle</u>:

<u>Ex</u>: $\tilde{M}_f = \tilde{F} \times \mathbb{R}$



$g_{\mathbb{H}^2} = \dfrac{4}{(1-|z|^2)^2} g_{\mathbb{E}^2}$

$\tilde{F} = \mathbb{H}^2 = \{|z| < 1\}$

$\partial \mathbb{H}^2 = S^1_\infty =$ equiv classes of geod rays $= P^1(\mathbb{R})$

Now

$\pi_1 M_f = \langle t, \pi_1 F \mid t g t^{-1} = f_*(g) \ \ \forall g \in \pi_1 F \rangle$

acts on $\tilde{M}_f$ with $\pi_1 F$ acting on $\tilde{F}$ pres. $\mathbb{R}$ factor, and $t \cdot (\tilde{x}, s) = (\tilde{f}^{-1}(x), s+1)$. So $\pi_1 M$ acts on $\bar{F} \times \mathbb{R}$ where $\bar{F} = \mathbb{H}^2 \cup S^1_\infty$. Projecting onto $\bar{F}$-factor, get action of $\pi_1 M$ on S^1_{univ}.

[Candel] If $\mathcal{F}$ is a taut fol of an irred atoroidal M^3, then $\exists$ a metric on M s.t. every leaf has const. curve -1.

<u>Cor</u>: $\tilde{\mathcal{F}}$ of $\tilde{M}$ is a fol by hyp. planes.

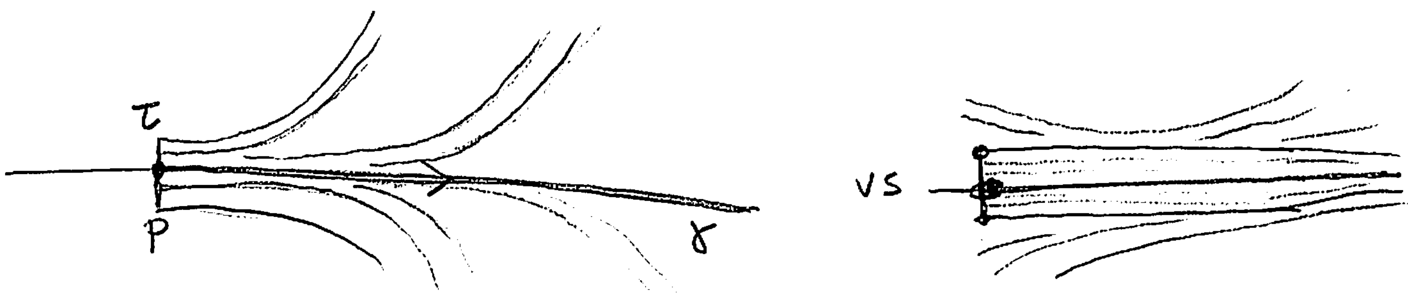
Want to identify the S^1_∞'s of these leaves together.

<u>Problem 1</u>: Non-Hausdorff leaf space

<u>Problem 2</u>: Even comparable pairs of leaves may not be uniformly close.

Suppose γ is a geodesic ray in $\tilde{L}$ starting at p.

Can we find a transv τ at p so that the hol of τ is defined along the whole of γ?



<u>Marker</u>: $I \times \mathbb{R}^+ \longrightarrow \tilde{M}$ where

a) each $t \times \mathbb{R}^+$ is a geod. ray in a leaf $\tilde{L} \cong \mathbb{H}^2$.

b) each $I \times s$ is a transv. to $\tilde{\mathcal{F}}$ of len $\leq \varepsilon_0$.

<u>Leaf Pocket Thm</u>: For every $\tilde{L}$ of $\tilde{\mathcal{F}}$, the endpts of markers are dense in $S^1_\infty(\tilde{L})$.

<u>Idea</u>: 1) Some leaf L of $\mathcal{F}$ has $\pi_1 L \neq 1$.

Otherwise, no holonomy $\Rightarrow$ transv. measure

$\Longrightarrow$ leaves have polynomial area growth,

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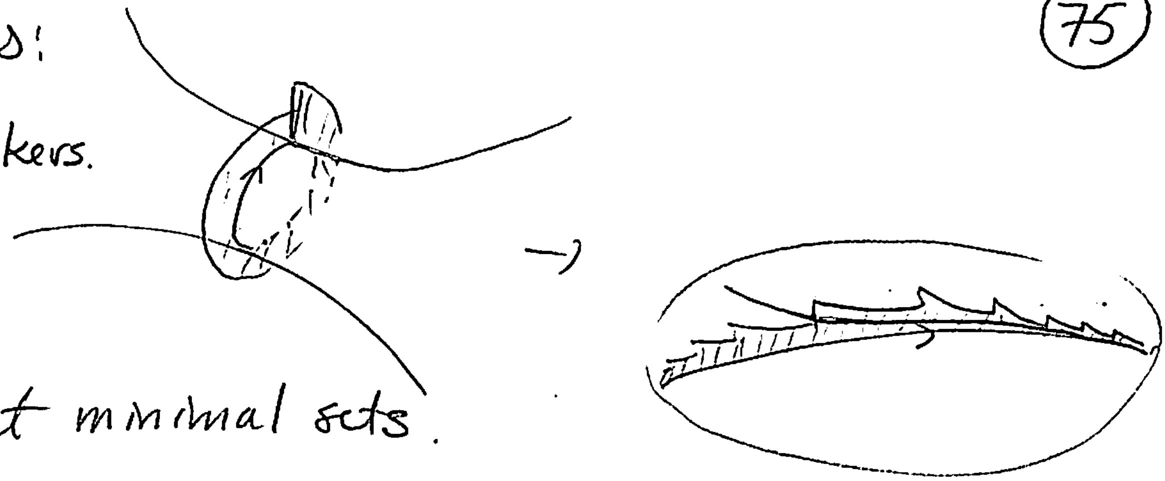
a contradiction since each leaf is $\mathbb{H}^2$ in this scenario.

2) Sawblades:

Lead to markers.



3) Argue about minimal sets.

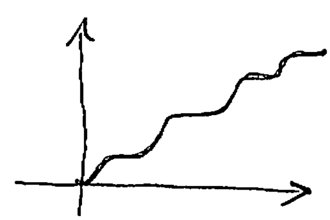
(e.g. suppose every leaf is dense.)

A <u>universal circle</u> S^1_{univ} for $\mathcal{F}$ is

a) An action of $\pi_1 M$ on S^1_{univ} $\left(\pi_1 M \xrightarrow[\rho_{univ}]{} \mathrm{Homeo}(S^1_{univ})\right)$

b) For every leaf $\tilde{L}$ a monotone map

$$\phi_{\tilde{L}} : S^1_{univ} \longrightarrow S^1_\infty(\tilde{L})$$

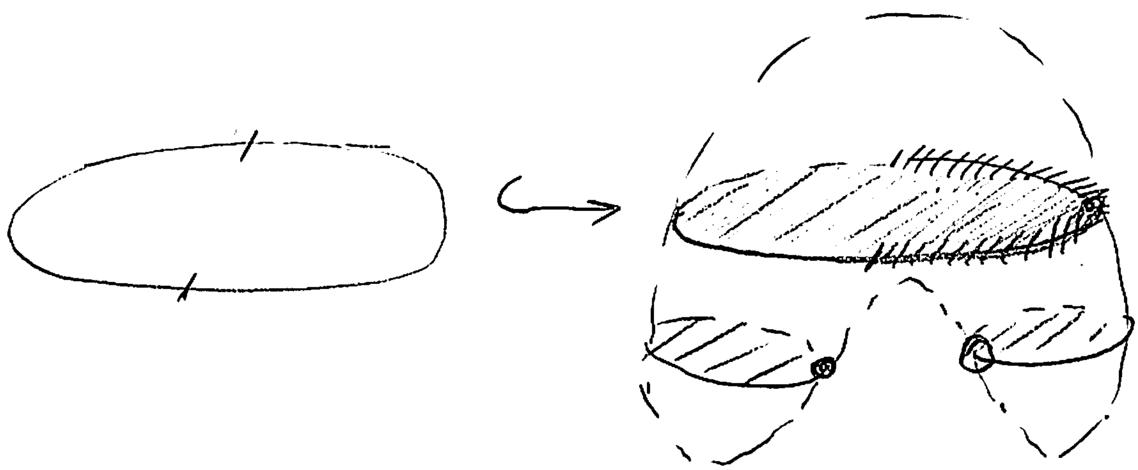


where $\forall\, \tilde{L}$ and $\alpha \in \pi_1 M$ one has

$$\begin{tikzcd}
S^1_{univ} \arrow[r, "\rho_{univ}(\alpha)"] \arrow[d, "\phi_{\tilde{L}}"'] \arrow[dr, phantom, "\circlearrowright"] & S^1_{univ} \arrow[d, "\phi_{\alpha(\tilde{L})}"] \\
S^1_\infty(\tilde{L}) \arrow[r, "\alpha"'] & S^1_\infty(\alpha(\tilde{L}))
\end{tikzcd}$$

<u>Thm:</u> $\exists$ taut fol of irred orient clsd M^3. Then $\mathcal{F}$ has a univ. circle. Moreover, the action of $\pi_1 M$ on S^1_{univ} is faithful $\Longrightarrow$ $\pi_1 M$ is a subgp of $\mathrm{Homeo}(S^1)$.

<u>Idea:</u> Use markers to take an inverse limit of the actions on the individual $S^1_\infty(\tilde{L})$.



For more, see Calegari's book or [Calegari-D 2003].