Geometric Topology: study of manifolds.

Convention: All manifolds will be orientable.

Homeo Prob: Given two closed n-manifolds M and N (say as simplicial complexes) are they homeomorphic?
Is this decidable? [i.e. $\exists$? a computer algorithm]

$n = 1$: Yes
$n = 2$: Yes
$n \geq 4$: No [Markov 1958] "manifolds are at least as complicated as finitely presented groups."
$n = 3$: Yes Focus of these lectures.

Role of Geometry:

$n = 2$: Any closed surface has a const. curve metric.
$n \geq 4$: Homogeneous geometry is "rare."
$n = 3$: Some have no const. curve metrics: $S^2 \times S^1$
Reason: Univ. cover not homeo to $S^3, E^3, H^3$

Geometrization Thm [Thurston, Perelman, ... ] Any closed 3-manifold has a decomposition along essential
$\natural$ and $\natural$ into pieces w/ metrics modelled
on one of: $S^3, E^3, H^3, S^2 \times R, H^3 \times R, N/1, S0V, SL_2R$
A $M^3$ is prime if $M = N_1 \# N_2 \Rightarrow$ some $N_i \cong S^3$.

Note: Any topological $M^3$ has a unique smooth structure. [So can read $\cong$ as homeo or diffeo.]

[Kneser-Milnor] Any closed $M^3 = N_1 \# \cdots \# N_k$ with $N_i$ prime. Unique up to perm. the $N_i$.

A closed surface $S \neq \emptyset$ in $M^3$ is incompressible/essential if $\pi_1 S \to \pi_1 M$.

[JSJ] Torus decomposition.

Most important/common geometry: $H^3$.

[Mflds w/ other geoms are classified...]

A hyperbolic structure on $M^3$ is a Riem. metric of const. curve $-1$. Equivalently, discrete, torsion free

$$M = \prod H^3$$

$$\Gamma = \pi_1 M \leq Isom^+(H^3)$$

$$M \cong \hat{\text{Mob}}^+(\hat{E})$$

$$\cong \text{PSL}_2 \mathbb{C}$$

Motivating example:

$\Lambda = \langle (1,2), (1,0) \rangle$

$H^2$

$\Lambda = \langle (1,2), (1,0) \rangle$

Not compact but area = $2\pi$
\[ \Gamma = \langle (1, 1), (2, -i, 1), (3, 2i, -1) \rangle \]

\[ M = \Gamma \backslash H^3 = S^3 \setminus \bigcirc \text{Borromean Rings} = G^3_2 \]

Volume \approx 7.327724753...

**Mostow Rigidity**: Suppose \( M \) and \( N \) are hyperbolic \( n \)-manifolds of finite volume where \( n \geq 3 \).
If \( \pi_1 M \cong \pi_1 N \) then \( M \) and \( N \) are isometric.

**Cor.**: Any geometric invariant of a hyp. \( n \)-fld for \( n \geq 3 \) is a topological invariant.

**Thurston's Mantra**: "Topology = Geometry" in dim 3.

How this connects to the homeo. problem, see [Kupersberg 2017].

1. Find the geom. decomp.
2. Non-hyp pieces are classified
3. For hyp. pieces need check for an isometry.
Much easier to check for isometries than homeos.

Enough theory, let's see this in practice!

SnapPy demo.