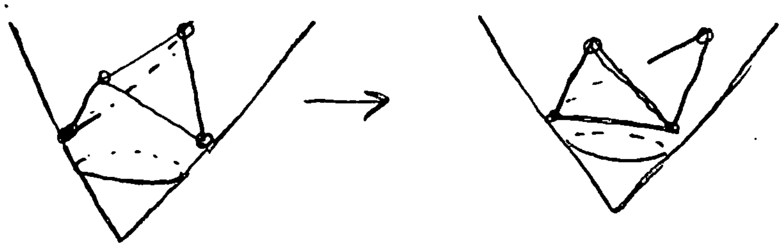
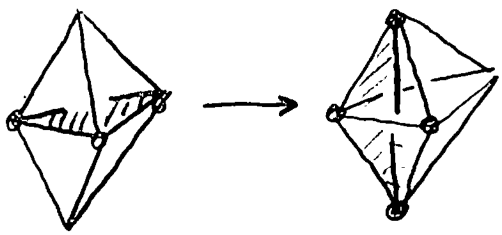
Lecture 10: Canonical cell. in 3D; closed 3-mflds

2D:



local test if given tri. is canon.

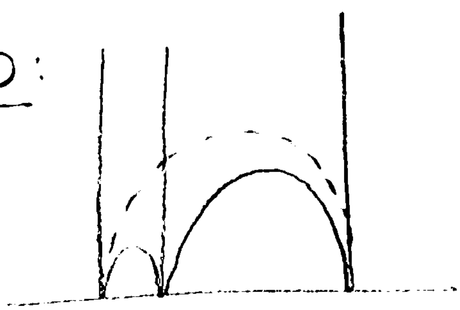
3D: Similar story, with flip replaced by $2 \to 3$ and $3 \to 2$ moves



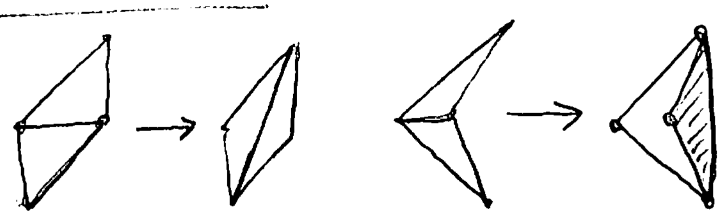
Still have a local test on faces and valence 3 edges, depending on whether things are concave/convex in $\mathbb{R}^{3,1}$ inside the light cone.

New issue: Creation of neg. orient tets

2D:

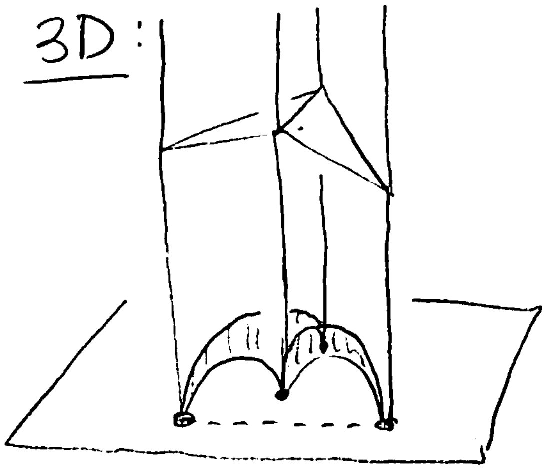


2D Euclidean:



copy into horosphere

3D:



Now the "move down" algorithm can get stuck.

In practice, the solution is to do a few random $2 \longleftrightarrow 3$ moves... [Why this works is mysterious.]

Solving the homeo. problem for M^3, N^3 that are finite vol. hyp with cusps.

1) Compute canon cell decomp $\mathcal{D}, \mathcal{J}$

2) M, N homeo. $\iff \mathcal{D}, \mathcal{J}$ are comb isomorphic.

Note: ($\Leftarrow$) clear, ($\Rightarrow$) is Mostow: $\pi_1 M \cong \pi_1 N \Rightarrow$ isometric.

Cor: $\mathrm{Isom}(M)$ = Comb. isom of $\mathcal{D}$, so $\mathrm{Isom}(M)$ is finite.

[What are some clsd mflds where Isom is infinite?]

What about $\mathrm{Diff}(M)$? Note $\mathrm{Isom}(M) \subseteq \mathrm{Diff}(M)$.

[Mostow says every diff. is homotopic to an isom.]

[Hatcher-Ivanov 1976] $\mathrm{Isom}(M) \hookrightarrow \mathrm{Diff}(M)$ is a homotopy equiv. In particular $\pi_0 \mathrm{Diff}(M) \cong \mathrm{Isom}(M)$.

[Gabai 2001] Same true for closed hyp M^3.

Sometimes called the Smale Conj.

[Smale] $O(3) \hookrightarrow \mathrm{Diff}(S^2)$ is a homotopy equiv.

[Hatcher 1983] $O(4) \hookrightarrow \mathrm{Diff}(S^3)$ is a homotopy equiv.

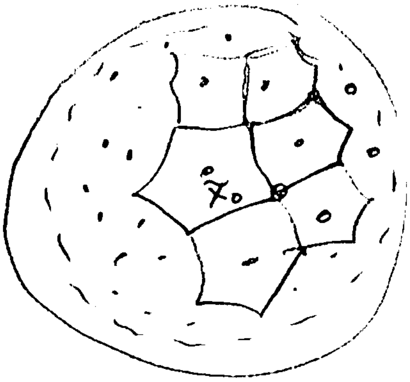
[New proof using Ricci flow see recent AMS notices art.]

—— // ——

M^3 closed hyp. Each conj. class in $\pi_1 M$ is rep. by a unique clsd geodesic. Also every clsd geod $\neq 1$ in $\pi_1 M$.

Pf: Consider action of $\gamma \in \pi_1 M$ on $\widetilde{M} = \mathbb{H}^3$. Now γ can't be elliptic since $\pi_1 M$ acts freely, and it can't be parabolic since such move pts in $\mathbb{H}^3$ arb. small amounts (and $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \cong B_\varepsilon(\mathbb{H}^3)$ for all $x \in M$). So γ is hyp with 2 f.p. on S^2_∞ and so preserves the geod $\tilde{g}$ joining them. ▨

Thm: M^3 closed hyp. Given $L > 0$, there are finitely many clsd geod of length $< L$.



Pf: Pick $\tilde{x}_0 \in \mathbb{H}^3$ and let $D_{\tilde{x}_0}$ be the coor. Dirichlet domain for $\pi_1 M \cdot \tilde{x}_0$.

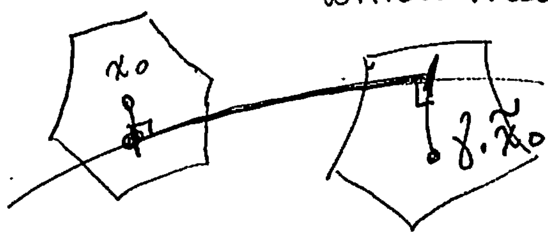
Claim: Every geod of len $< L$.

can be rep by $\gamma \in \pi_1 M$ with

$$\mathrm{dist}(\tilde{x}_0, \gamma \tilde{x}_0) < L + 2\,\mathrm{diam}(D_{\tilde{x}_0})$$

[This is enough since $\pi_1 M \cdot \tilde{x}_0$ is discrete.]

Pf of Claim: Pick a lift $\tilde{g}$ of a geod g of len $< L$ which meets $D_{\tilde{x}_0}$; γ in $\pi_1 M$ translates by $\mathrm{len}(g)$ along γ. ▨

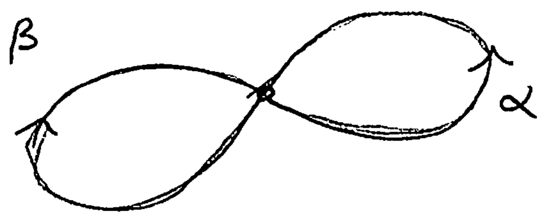


[Margulis] $\{\#$ primitive closed geod of len $< L\} \sim e^{2L}/2L$

A systole of M is a shortest closed geod.

Lemma: A systole is an emb. S^1

Pf: If the sys. has a self-intersection then at least one of α, β is $\neq 1$ in $\pi_1 M$. Say $\alpha = 1$. Then α has a cor. closed geod h with $\mathrm{len}(h) < \mathrm{len}(\alpha) < \mathrm{len}(\gamma)$, a contradiction. ▨



$\gamma = \alpha * \beta$

Thm: Let g be any emb. clsd geod in a clsd hyp M^3. Then $M \setminus g$ has a comp. hyp metric of finite vol.

Pf [Next time].

Solving Homeo Prob for clsd hyp M, N.

1) Find all systols for M, N.
2) Find all isom $(M \setminus sys)$ to $(N \setminus sys)$
3) See if any in 2 extend to M, N (Dehn filling).