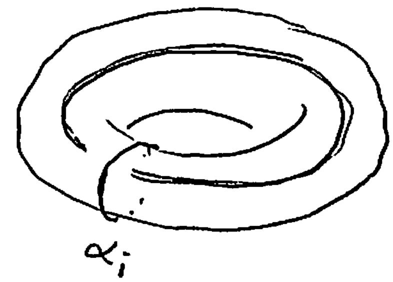
Lecture 12: Hyperbolic Dehn filling.

Solving Homeo Prob for clsd hyp A^3, B^3.

1) Find all systoles for A and B.



2) Find all isom between the hyp mflds $M_i := (A \setminus sys_i)$ and $N_j = (B \setminus sys_j)$

3) Check if any isom from (2) extends to a homeo $A \to B$.

Point of (3) is Dehn filling. If $X_i = M_i \setminus$ cusp nbhd and $Y_j = N_j \setminus$ cusp nbhd, then

$$A = X_i(\alpha_i) = X_i \cup (D^2 \times S^1) \quad \text{s.t. } \alpha_i \leftrightarrow \partial D^2 \times \{pt\}$$

$$B = Y_j(\beta_j)$$

So question is does some isom $M_i \to N_j$ send $\alpha_i \to \beta_j$.

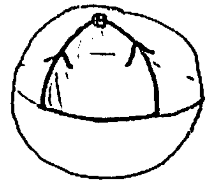
Def: A slope on a torus is an isotopy cls of ess. simple closed curves.

[Cor to the prim elts of $H_1(T;\mathbb{Z})$ mod. ± 1]

Cone manifolds of const. curve:

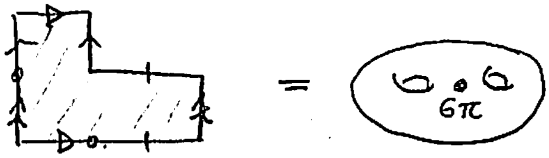
2D: Isolated cone sing:

$\mathbb{E}^2$



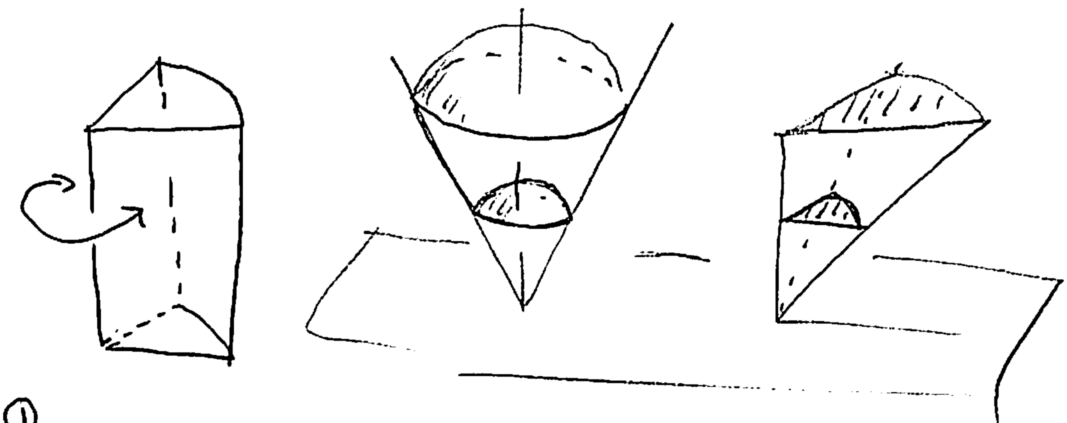
S^2

Allow cone angle $\geq 2\pi$.



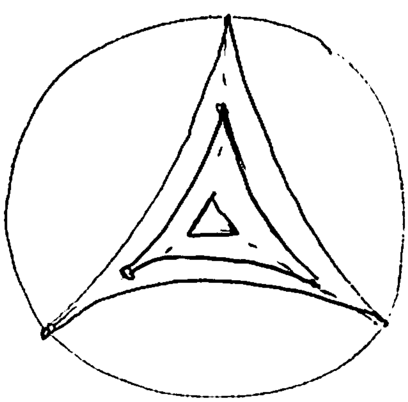
When all cone angles are $\frac{2\pi}{n}$ for $n \in \mathbb{N}$, have an orbifold.

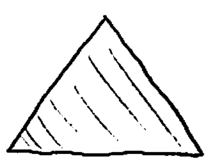
3D: $2D \times S^1$



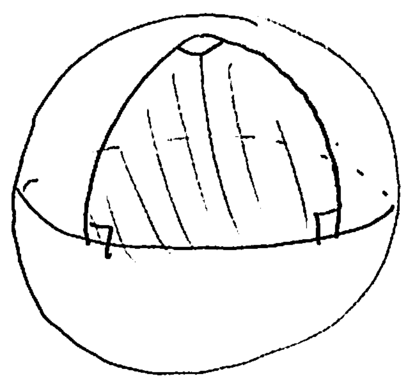
Key examples: (1) For $\theta \in [0, \pi]$, $\exists$ a unique equalateral tri T_θ with angle θ.

$\theta < \pi/3$ $\qquad$ $\theta = \pi/3$ $\qquad$ $\theta > \pi/3$

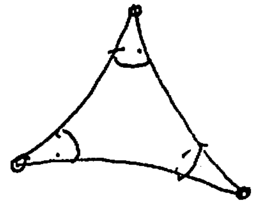




$\mathbb{E}^2$



S_θ = double of T_θ



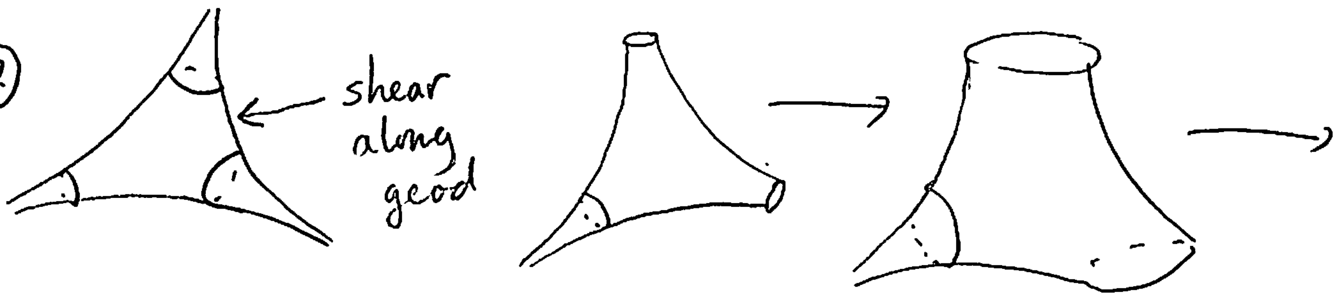
$\theta = 0$ $\quad S_\theta = S^2 \setminus 3$pts

$\theta > 0$ $\quad S_\theta = S^2$ with 3 cone pts of angle 2θ.

As $\theta \nearrow \pi/3$, $\mathrm{diam}(S_\theta) \to 0$

If rescale metric to keep $\mathrm{diam}(S_\theta) = 1$, then conv. to $S_{\pi/3}$ as metric spaces. Same as $\theta \searrow \pi/3$.

2



$\mathcal{T}(S^2 \setminus 3\text{pts}) = \{pt\}$

Perturb metric, complete adds a geod.

<u>3D</u>: [Combine 1 and 2] Despite Mostow, deformations exist by "parameter count".

If $\mathcal{T}$ has t tets and k cusps, then can reduce to just $t-k$ gluing eqns. (<u>not</u> incl. cusp eqns). So $\dim_{\mathbb{C}}(z_i \text{ solv. eqns}) = k$.

Generic completion adds one pt per cusp.

However, given a slope α in a cusp there always metrics g_θ for $\theta \in [0, \varepsilon]$ on $M(\alpha)$ with cone sing. along the core of $D^2 \times S^1$ with angle θ, where $g_0 =$ complete hyp metric on M.

If $\varepsilon \geq 2\pi$, then $M(\alpha)$ is hyp. and say $M(\alpha)$ is obtained by <u>hyperbolic Dehn filling</u>.

<u>Ex</u>: $M = S^3 \setminus$ 

Here g_θ is a metric on S^3 sing. along .

$\alpha =$  meridian

Have hyp g_θ for $\theta \in [0, 2\pi/3)$ the collaspe to a Euclid. orbifold. Said orb. is also a limit of spherical cone metrics, e.g. (S^3, g_π) is a 2-fold quotient of $L(5,2)$

[Hodgson-Kerkhoff 2008] Suppose M^3 finite vol hyp and T a horotorus cross section of its only cusp. If α is a slope on T with

$$\hat{L}(\alpha) = \frac{L(\alpha)}{\sqrt{\text{Area}(T)}} \geq 7.58$$

then $M(\alpha)$ can be obtained by hyp. Dehn filling on M.

Moreover, the core geod in $M(\alpha)$ has len ≤ 0.162

Cor: All but at most 60 $M(\alpha)$ are hyperbolic.

[HK] M^3 clsd hyp with γ a simple closed geod of len ≤ 0.162. Then $\exists$ hyp. cone metrics g_θ on (M, γ) with $\theta \in [0, 2\pi]$ connecting the complete hyp str on $M \setminus \gamma$ to that on M.

Real key is not length of geod but depth of tubes...

Local param, harmonic 1-forms w/ twist coeffs...