Lecture 12: Hyperbolic Dehn filling.

Solving Homeo Prob for cld hyp \( A^3, B^3 \).

1) Find all systoles for \( A \) and \( B \).
2) Find all isom between the hyp manifds \( M_i = (A \setminus \text{sys}_i) \) and \( N_j = (B \setminus \text{sys}_j) \).
3) Check if any isom from (2) extends to a homeo \( A \rightarrow B \).

Point of (3) is Dehn filling. If \( X_i = M_i \setminus \text{cusp nbhd} \) and \( Y_j = N_j \setminus \text{cusp nbhd} \), then

\[
\begin{align*}
A &= X_i(\alpha_i) = X_i \cup (D^2 \times S^1) \text{ s.t. } \alpha_i \leftrightarrow 2D^2 \times \text{pt} \\
B &= Y_j(\beta_j)
\end{align*}
\]

So question is does some isom \( M_i \rightarrow N_j \) send \( \alpha_i \rightarrow \beta_j \).

**Def:** A slope on a torus is an isotopy cls of ess. simple closed curves.

\([\text{cor to the prim elts of } H_1(T; \mathbb{Z}) \text{ mod. } \pm 1]\)
Cone manifolds of constant curve:

2D: Isolated cone sing: \( \Delta = \bigcirc \)

Allow cone angle \( \geq 2\pi \).

When all cone angles are \( \frac{2\pi}{n} \) for \( n \in \mathbb{N} \), have an orbifold.

3D: \( 2D \times S^1 \)

Key examples: For \( \theta \in [0, \pi] \), \( \exists \) a unique equilateral tri \( T_\theta \) with angle \( \theta \).

\( \theta < \pi/3 \) \( \theta = \pi/3 \) \( \theta > \pi/3 \)
$S_{\theta} = \text{double of } T_{\theta}$

$\theta = 0 \quad S_{\theta} = S^2 \setminus 3 \text{pts}$

$\theta > 0 \quad S_{\theta} = S^2 \text{ with 3 cone pts off angle } 2\theta.$

As $\theta \to \pi/3$, $\text{diam}(S_{\theta}) \to 0$

If rescale metric to keep $\text{diam}(S_{\theta}) = 1$, then conv. to $S_{\pi/3}$ as metric spaces. Same as $\theta \to \pi/3$.

$\mathcal{T}(S^2 \setminus 3 \text{pts}) = \mathcal{F} \setminus 3 \text{pts}$

Perturb metric, complete adds a geod.

3D: [Combine 0 and 2] Despite Mostow, deformations exist by "parameter count".

If $\mathcal{T}$ has $t$ tets and $k$ cups, then can reduce to just $t-k$ gluing eqns. (not incl. cup eqns). So $\dim_{\mathbb{C}}(\mathcal{T}; \text{solv. eqns}) = k$.

Generic completion adds one pt per cusp.

However, given a slope $\alpha$ in a cusp
there always metrics $g_\theta$ for $\theta \in [0, \pi]$ on $M(\alpha)$ with cone sing. along the core of $S^3 \times S^1$ with angle $\theta$, where $g_0 =$ complete hyp metric on $M$.

If $\pi \geq \frac{2\pi}{3}$, then $M(\alpha)$ is hyp. and say $M(\alpha)$ is obtained by hyperbolic Dehn filling.

Ex: $M = S^3 \setminus \{ \text{sing. along } S^2 \}$

$\alpha = \{ \text{meridian} \}$

Here $g_\theta$ is a metric on $S^3$ sing. along $S^2$.

Have hyp $g_\theta$ for $\theta \in [0, \frac{2\pi}{3})$ the collapse to a Euclid.

orbifold. Said orb is also a limit of spherical cone metrics, e.g. $(S^3, g_{\pi})$ is a 2-fold quotient of $L(5,2)$

[Hodgson-Kerckhoff 2008] Suppose $M^3$ finite vol hyp and $T$ a horotorus cross section of its only cuusp.

If $\alpha$ is a slope on $T$ with

$$\hat{L}(\alpha) = \frac{L(\alpha)}{\sqrt{\text{Area}(T)}} \geq 7.58$$

then $M(\alpha)$ can be obtained by hyp. Dehn filling on $M$. 

Moreover, the core geod in $M(\alpha)$ has len $\leq 0.162$

Cor: All but at most 60 $M(\alpha)$ are hyperbolic.

[HK] $M^3$ clsd hyp with $\gamma$ a simple closed
good of len $\leq 0.162$. Then $\exists$ hyp. cone metrics
g $\theta$ on $(M, \gamma)$ with $\theta \in [0, 2\pi]$ connecting
the complete hyp str on $M \setminus \gamma$ to that on $M$.

Real key is not length of geod but depth of
tubes...

Local param, harmoniz 1-forms w/ twist coeffs...