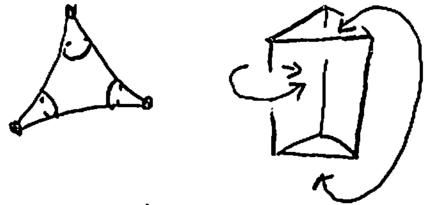
Lecture 13: More on hyperbolic Dehn filling

Cone mflds:  $(S^3,$  $^{2\pi/3})$ is an $\mathbb{E}^3$ cone mfld.

Have both a cone angle and a twist along the sing. locus. 

Dehn filling: M with $\partial M =$ , α slope in ∂M $\Big\} \longrightarrow M(\alpha) = M \cup D^2 \times S^1$

$\alpha \leftrightarrow \partial D^2 \times pt$

Hyperbolic Dehn filling: $\exists$ a family of hyperbolic cone metrics g_θ on $M(\alpha)$ for $\theta \in [0, 2\pi]$, singular along the core $pt \times S^1$ with cone angle θ.

Here, $\theta = 0$ cor. to the complete hyp str on M.

[Hodgson - Kerckhoff] g_θ is unique when it exists.

Note: If $M(\alpha)$ is a hyp. Dehn filling of M, then the core of the added solid torus is a clsd geod.

Ex: N any clsd hyp 3-mfld. $\exists\, K : S^1 \hookrightarrow N$ such that a) $M = N \setminus K$ is hyperbolic.

b) K is null homotopic.

Then N is not obtained by hyp. Dehn fill on M!

(2)

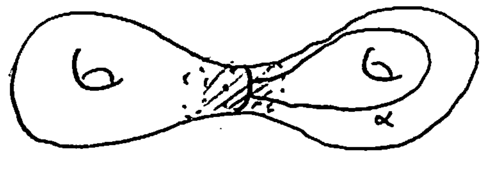
[Hodgson-Kerckhoff 2008] Suppose M^3 is finite-vol hyp with T a horotorus cross section of its only cusp. If α is a slope on T with $\hat{L}(\alpha) = \dfrac{\text{len}(\alpha)}{\sqrt{\text{Area}(T)}} \geq 7.58$ then $M(\alpha)$ can be obtained by hyp. Dehn filling on M. Moreover the core geod has len ≤ 0.162.

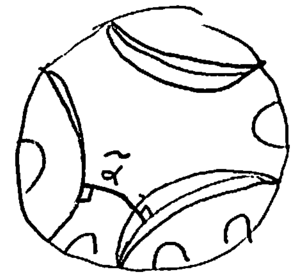
<u>Cor</u>: All but at most 60 $M(\alpha)$ are hyp. Dehn fillings on M.

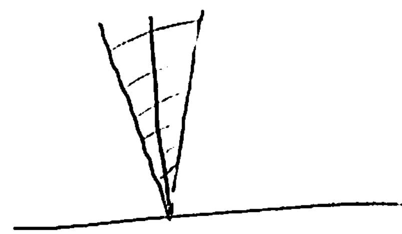
[Lackenby-Meyerhoff 2010] All but at most 10 $M(\alpha)$ are hyperbolic.

[HK] N^3 clsd hyp with γ a simple clsd geod of len ≤ 0.162. Then N can be obtained by hyp. Dehn surgery on $M = N \setminus \gamma$.

Real key is not len of geod. but the depth of tubes. Suppose γ is a simple clsd geod. For small r, $B_r(\gamma)$ is an open solid torus, a "banana quotient."







tube radius $\gamma = \sup \{ r \mid B_r(\gamma) \text{ is embedded} \}$

$$= \frac{1}{2} \min \{ \operatorname{len}(\alpha) \mid \underbrace{\alpha \text{ a geod arc w/ end pts on } \gamma}_{\alpha \not\subset \gamma} \}$$

Makes sense for the sing. locus in a cone mfld. A key part of [HK]'s analysis is that one can always increase the cone angle θ by an amount dep. on tube rad (core).

[Recall collaspe to Euclidean cone mflds...]

In particular a collasping fam of hyp cone structures must collaspe near the cone locus.

[Back to example on page 1.] Need to understand harmonic 1-forms with twisted coeffs (by adop acting on $\underline{sl}_2$). When $\hat{L}(\alpha) \geq 7.58$, for small θ the initial tube has large radius, and can't collaspe until $\theta \geq 2\pi$.

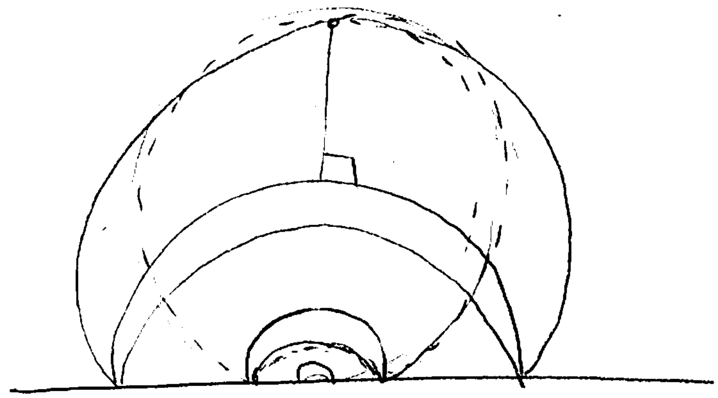
[Gabai-Meyerhoff-N. Thurston 2003; Gabai-Trnkova 2015]

With one exception, every clsd hyp 3-mfld has a simple clsd geod γ with tube rad $\geq \frac{\log(3)}{2} \geq 0.549$

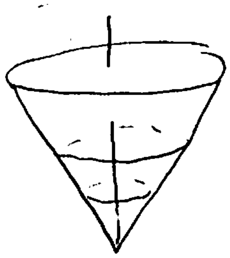
Exception is Vol3, the third smallest clsd hyp 3-mfld.

Key tool in Gabai's proof that $\mathrm{Diff}(M) \simeq_{h.e.} \mathrm{Isom}(M)$ for hyp M^3.

Tubes and cusps are friends:



The boundary of a tube has a Euclid. metric.

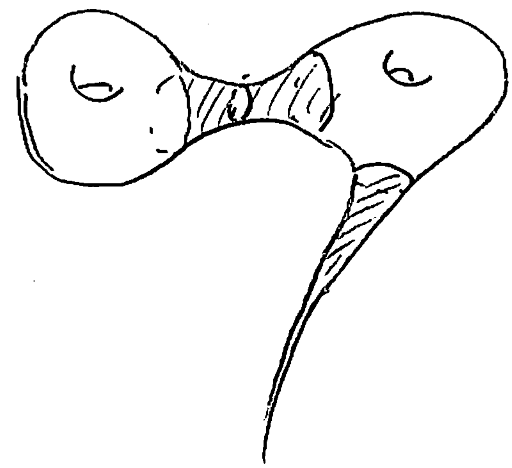


Horoballs are limits of tubes.

Cusp nbhds are $(\text{Horoball})/\mathbb{Z}^2$ $\quad \partial(\text{H.ball}) \cong \mathbb{R}^2$

Tube nbhds are $(\text{tube})/\mathbb{Z}$ $\quad \partial(\text{tube}) \cong S^1 \times \mathbb{R}$.

<u>Margulis Lemma:</u> $\exists\, \varepsilon_n > 0$ such that for all finite vol hyp n-mflds M the set



$$M_{thn,\varepsilon} = \{x \in M \mid inj_x M < \varepsilon\}$$

is a finite union of cusp nbhds and emb. tubes about simple clsd geod of length $\leq \varepsilon_n$

<u>Idea:</u> Non-commuting isom γ_1, γ_2 that move same pt $p \in \mathbb{H}^n$ by small amounts will generate an indiscrete gp.

<u>Quatative Version:</u> The shorter γ is the larger its tube radius