Lecture 6: Hyperboloid model

Goal: M^3 comp hyp with cusps $\rightsquigarrow$ cannonical ideal "triangulation" } Use to solv. homeom prob.

Geometry of S^2: $\langle x, y\rangle = x^t I y = x_1 y_1 + x_2 y_2 + x_3 y_3$

$S^2 = \{x \in \mathbb{R}^3 \mid \langle x, x\rangle = 1\}$ has a Riem. metric by rest. $\langle\, ,\, \rangle$ to each $T_p S^2$.

Claim: $\mathrm{Isom}(S^2) = O(3) = \{A \in GL_3\mathbb{R} \mid \langle Ax, Ay\rangle = \langle x, y\rangle$ for all x, y, i.e. $A^tA = 1\}$

Pf: Clearly $O(3) \subseteq \mathrm{Isom}(S^2)$. Now $O(3)$ acts trans. on S^2: Any $p \in S^2$ can be moved to $\{x_2 = 0\}$.

by $\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$ which is rotation about the z-axis

and then to $e_3 = (0,0,1)$ by $\begin{pmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{pmatrix}$.

Moreover, the map $\begin{pmatrix} \text{Stab of } e_3 \\ \text{in } O(3) \end{pmatrix} \longrightarrow O(T_{e_3} S^2 = \langle e_1, e_2\rangle)$ is an isom. Thus given any isom f, $\exists\, A \in O(3)$ s.t. $A \circ f$ fixes e_3 and acts triv on $T_{e_3} S^2$.

By uniqueness of geodesics and the fact that the exponential map is onto, $A \circ f$ is id_{S^2}.

So $f = A^{-1}$ and $\mathrm{Isom}(S^2) = O(3)$. $\blacksquare$

Claim: The geodesics on S^2 are great circles, the intersection of S^2 with a plane through 0.

Pf: Enough to show for $P = \{z = 0\}$ since $O(3)$ acts trans on such planes (as it does so on their normal vecs).

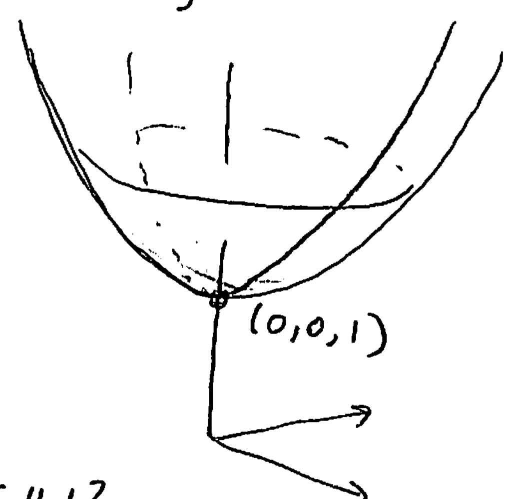
Now P is fixed by $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and let γ be unique geod. with $\gamma'(0) = (0,1,0) \in T_{(1,0,0)} S^2$.

Since A fixes $\gamma'(0)$, it fixes γ pointwise $\Rightarrow \gamma = P \cap S^2$. $\square$

Geometry of $\mathbb{H}^2$: $\langle x, y \rangle = x^t J y$ for $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$$= x_1 y_1 + x_2 y_2 - x_3 y_3$$

$$\mathbb{H}^2 = \left\{ x \in \mathbb{R}^3 \;\middle|\; \langle x, x \rangle = -1 \iff x_1^2 + x_2^2 = x_3^2 - 1 \text{ and } x_3 > 0. \right\}$$



Set $O(2,1) = \left\{ A \in GL_3\mathbb{R} \;\middle|\; \begin{array}{l} A \text{ pres } \langle\, , \rangle \\ \text{i.e. } A^t J A = J \end{array} \right\}$

which pres $\{\langle x, x \rangle = -1\}$. Let $O_0(2,1)$ be the subgp pres $\mathbb{H}^2$.

Claim: The rest of $\langle\, , \rangle$ to each $T_p \mathbb{H}^2$ is pos. def, making it a Riem. mfld.

Moreover $\mathrm{Isom}(\mathbb{H}^2) = O_0(2,1)$.

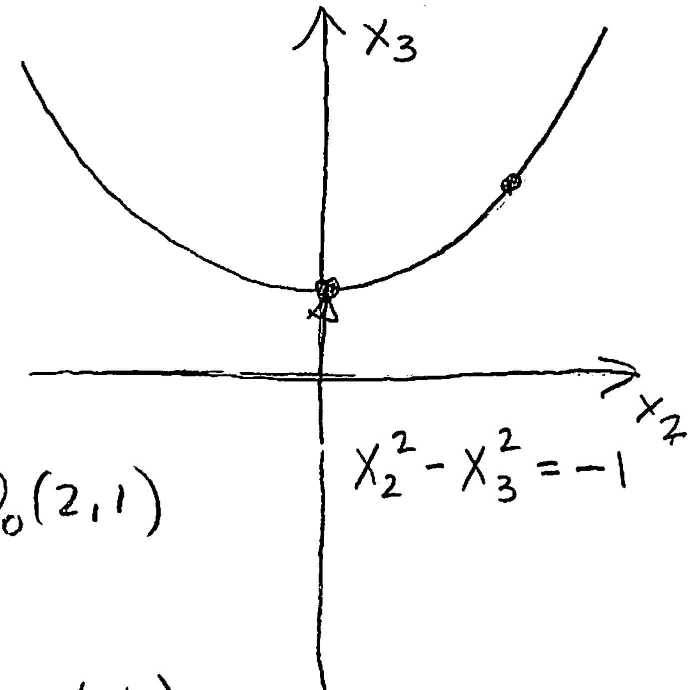
Pf: $O_0(2,1)$ acts trans on $\mathbb{H}^2$ since

a) $O_0(2,1) \supseteq \left\{ \left(\begin{array}{c|c} O(2) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 \quad 0 & 1 \end{array} \right) \right\}$ so any $p \in \mathbb{H}^2$ can be rotated so that $x_1 = 0$

b) Param the hyp $x_1 = 0$

by $x_2 = \sinh(t) = \frac{1}{2}(e^t - e^{-t})$

and $x_3 = \cosh(t) = \frac{1}{2}(e^t + e^{-t})$



Then $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix}$ is in $O_0(2,1)$

and takes e_3 to $(0, \sinh t, \cosh t)$.

So $\langle , \rangle|_{T_p\mathbb{H}^2}$ is pos. def since $\langle , \rangle|_{T_{e_3}\mathbb{H}^3} = x_1^2 + x_2^2$

The rest is the same as for S^2. ∎

Again, if P is a plane through 0, then $P \cap \mathbb{H}^2$ is a geodesic when $\neq \emptyset$. Also, $\gamma(t) = (0, \sinh t, \cosh t)$ is a unit speed geod:

$\gamma' = (0, \cosh t, \sinh t) \Rightarrow \langle \gamma', \gamma' \rangle = \cosh^2 t - \sinh^2 t = 1$.

<u>Claim</u>: $\mathrm{dist}_{\mathbb{H}^2}(x,y) = \cosh^{-1}(-\langle x,y\rangle)$

Cf: $\mathrm{dist}_{S^2}(x,y) = \cos^{-1}(\langle x,y\rangle)$.

<u>Pf</u>: Using A in $O_0(2,1)$, can assume $x = e_3$ and y has $y_1 = 0$ and $y_2 > 0$. Then $-\langle x,y\rangle = +y_3 = \cosh(t)$ where $t = \mathrm{dist}_{\mathbb{H}^2}$ as γ is unit-speed. ▨

<u>Comments</u>: 1) To define $\mathbb{H}^n$, use $\mathbb{R}^{n,1}$ with $\langle x,y\rangle = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1}$. Note $\mathbb{H}^n$ has many $\mathbb{H}^k$'s in it — just intersect with a subsp. through 0.

2) $\mathrm{Isom}^+(\mathbb{H}^n) = SO_0(n,1)$.

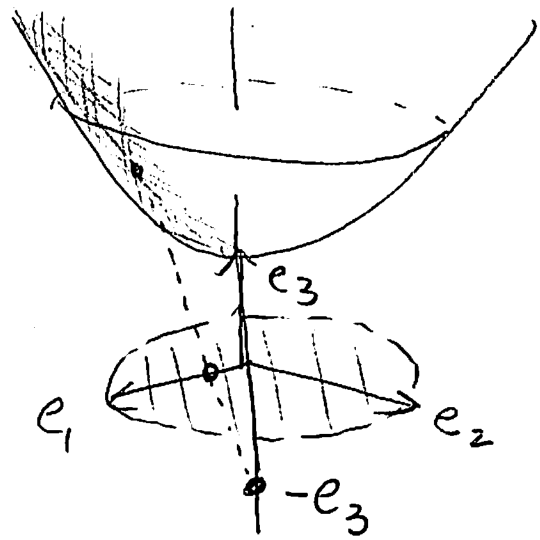
Also circum. of circle of radius t is $2\pi\sinh t$. $\approx \pi e^t$ for large t!

3) $\Gamma\backslash\mathbb{H}^n$ for $\Gamma = SO_0(n,1) \cap SL_{n+1}\mathbb{Z}$ gives a hyp. orbifold with finite volume and cusps.

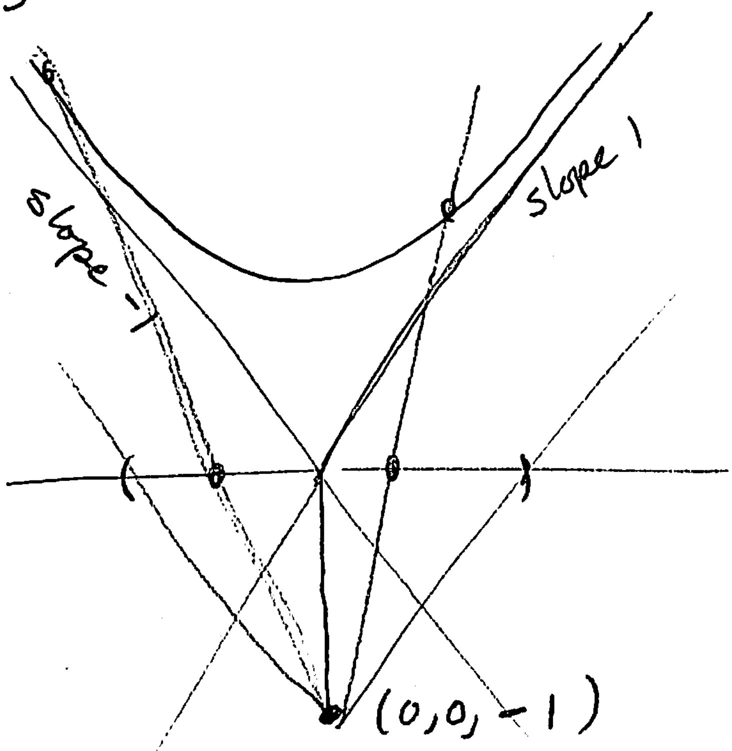
4) Great for computation — its just matrices!

5) Also, $\mathbb{H}^n$ is complete (as a metric space $\Leftrightarrow$ as a Riem. manifold.) as closed balls are cpt.

Back to $n=2$: How does this connect to the Poincaré model?



Stereographically project onto open unit disc in $x_3 = 0$ plane towards $-e_3$.



Not so hard to write out and calc. that this is an isom...