Lecture 7: Canonical cell decompositions.

Previously... \( \mathbb{R}^{n+1}_n = \mathbb{R}^{n+1}_n \) with \( \langle x, y \rangle = x^t J y \) for \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

\( H^n = \{ x \in \mathbb{R}^{n+1}_n \mid \langle x, x \rangle = -1 \text{ and } x_{n+1} > 0 \} \)

with R-metric = \( \langle \cdot , \cdot \rangle \mid T_p H^n \). Then \( \text{Isom}^+(H^n) \) is

\( \text{SO}_0(n, 1) = \{ A \in \text{SL}_n(\mathbb{R}) \mid A^t J A = J \text{ and } A \text{ pres } H^n \} \)

Geodesics are ("plane through 0") \( \cap H^n \).

Connections to other models:

Poincaré disk:

\( D = \{ (x_1, x_2, 0) \mid x_1^2 + x_2^2 < 1 \} \)

Use Stereographic projection:

\( H^2 \rightarrow D \quad (x_1, x_2, x_3) \mapsto \frac{1}{1 + x_3} (x_1, x_2, 0) \)

Can check this is an isom.

Note: \( H^n \) has const curve as it is isotropic. The constant is \(-1\) as:

\[
\text{len}(\text{circle of radius } r) = 2\pi \sinh r = 2\pi \left( r + \frac{1}{6} r^3 + O(r^5) \right) = 2\pi \left( r - \frac{K}{6} r^3 + O(r^4) \right)
\]
Klein model:

\[ K = \{(x_1, x_2, 1) | x_1^2 + x_2^2 < 1\} \]

Stereo proj from 0
\[(x_1, x_2, x_3) \rightarrow \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1\right)\]

Geodesics are Euclidean straight lines, isometries are projective transformations.

Sphere at infinity:

\[ L = \text{light cone} = \{ x \in \mathbb{R}^{n+1} | \langle x, x \rangle = 0 \} \]

\[ S^n_{\infty} = \{ x \in L | x_{n+1} > 0 \} / \mathbb{R}_{>0} \]

Horoball: \( v \in L \) with \( x_{n+1} > 0 \).

Set \( H_v = \{ x \in H^n | -1 \leq \langle v, x \rangle \} \)
and \( S_v = \{ x \in H^n | -1 = \langle v, x \rangle \} \)
eqn of a plane.

\[ \langle x, v \rangle = x_1 - x_3 = -1 \]
In 3D:

Curve is a parabola:

\[(t^2, \sqrt{2} t, t^2 + 1)\]

for \(v = (1, 0, 1)\)

3 Kinds of Orbit (for any \(H^n\)):

- **Elliptic**: fix a pt in \(H^n\), e.g. \(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\)
- **Hyperbolic**: no fixed pt in \(H^n\), two on \(S^{n-1}\), e.g. \(\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix}\)
  
  Have an axis, etc.
- **Parabolic**: no fixed pt in \(H^n\), one on \(S^{n-1}\), e.g. \(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & -t & 1 \end{pmatrix}\)

A horosphere is the orbit of a pt under the parab. that fix a pt on \(S^{n-1}\).

E.g. \(A_+ = \begin{pmatrix} 1-t^2 & \sqrt{2}t & t^2 \\ -\sqrt{2}t & 1 & \sqrt{2}t \\ -t^2 & \sqrt{2}t & 1+t^2 \end{pmatrix}\) in \(SO_0(2,1)\).

Are the parabolics fixing \(v = (1,0,1)\) and \(S_v\) is the orbit of \(e_3\) under \(A_+\).
Suppose $S$ is a hyperbolic surface of finite area with cusps.
Choose disjoint cusp neighborhoods with equal area.

Gives a $\pi_1 S$-equivariant collection of horoballs.

Let $V$ be collection of vectors on the light cone corresponding to these. Note that $\forall C, \exists u \in V | u_3 \leq C^2$
if $V$ accumulates somewhere, can assume its at $(1,0,1)$ and then all $w, w' \in \mathbb{L}$
are near $V$ have $H_w \cap H_{w'} \neq \emptyset$ which violates that cusp neighborhoods are disjoint. (Note: small $v \in \mathbb{L} \iff$
large horoball).

Let $P$ be the (ordinary) convex hull of $V$ in $\mathbb{R}^3$, and now project $\partial P$ onto $H^2$ via lines through $O$. The result is a dec omp $D$ of $H^2$ into ideal polyhedra, since the edges in the cellulation of $\partial P$ turn into geodesics.
Note that our \( \pi_1 S \) action pures \( V \) and hence \( D \). Moreover, no elt \( \gamma \in \pi_1 S \) sends a cell \( C \) of \( \dim > 0 \) of \( D \) to itself. If \( C \) comes from \( v_1, v_2, \ldots, v_k \) in \( V \), then \( v_{\beta} = \frac{1}{k}(v_1 + \cdots + v_k) \) is fixed by \( \gamma \). and hence \( \frac{v_{\beta}}{\sqrt{-\langle v_{\beta}, v_{\beta} \rangle}} \) in \( H^2 \) is fixed by \( \gamma \), contradicting that \( \pi_1 S \) acts freely.

Thus \( D \) descends to a cellulation \( \overline{D} \) of \( S \), the canonical Epstein-Penner decomposition.