1. Find a primitive generator for \( \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) \) over \( \mathbb{Q} \). Be sure to justify your answer.

2. Let \( K/F \) be a Galois extension and \( \alpha \in K \). Define the norm of \( \alpha \) from \( K \) to \( F \) to be

\[
N_{K/F}(\alpha) = \prod_{\sigma \in \text{Gal}(K/F)} \sigma(\alpha)
\]

(a) Prove that \( N_{K/F}(\alpha) \in F \).

(b) Prove that \( N_{K/F}(\alpha \beta) = N_{K/F}(\alpha)N_{K/F}(\beta) \). Thus, the norm gives a group homomorphism \( K^\times \to F^\times \).

(c) Prove that \( N_{K/F}(a\alpha) = a^n N_{K/F}(\alpha) \) where \( a \in F \) and \( n = [K:F] \).

(d) Let \( K = F(\sqrt{D}) \) be a quadratic extension of \( F \). Show that \( N_{K/F}(a + b\sqrt{D}) = a^2 - Db^2 \), where \( a, b \in F \).

(e) Let \( K \) be the splitting field of \( x^3 - 2 \) over \( \mathbb{Q} \). Compute \( N_{K/Q}(\alpha) \) for \( \alpha = \sqrt[3]{2} \) and for \( \zeta = \zeta_3 \).

Note: It is possible to define \( N_{K/F}(\alpha) \) even when \( K/F \) is not Galois, see Problem #17 in Section 14.2 of our text.

3. As in the last problem, let \( K/F \) be a Galois extension with \( n = [K:F] \) and \( \alpha \in K \).

(a) Let \( f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 \) in \( F[x] \) be the minimal polynomial of \( \alpha \) over \( F \). Prove that \( d \) divides \( n \) and that there are \( d \) distinct Galois conjugates of \( \alpha \) in \( K \), each of which is repeated \( n/d \) times in the formula for \( N_{K/F}(\alpha) \). Use this to show \( N_{K/F}(\alpha) = (-1)^n a_0^{n/d} \).

(b) For \( \alpha \in K \), consider \( T_\alpha: K \to K \) where \( T_\alpha(\beta) = \alpha \beta \). As you know, this is an \( F \)-linear transformation; let \( A \) be the associated matrix with respect to some \( F \)-basis of \( K \). If \( K = F(\alpha) \), show that \( \det(A) = N_{K/F}(\alpha) \). (In fact, this is true with no assumption on \( \alpha \).)

4. Consider \( f(x) = (x^3 - 2)(x^3 - 3) \) in \( \mathbb{Q}[x] \).

(a) Determine the Galois group of \( f(x) \) over \( \mathbb{Q} \). That is, if \( K \) is the splitting field of \( f \), compute \( \text{Gal}(K/\mathbb{Q}) \). Note: You may assume that \( \mathbb{Q}(\sqrt[3]{2}) \) and \( \mathbb{Q}(\sqrt[3]{3}) \) are distinct subfields of \( K \). (This can be verified by checking that \( \sqrt[3]{3} - \sqrt[3]{2} \) is a root of \( x^9 - 3x^6 + 165x^3 - 1 \) and that the latter polynomial is irreducible.)

(b) Find all subfields of \( K \) that contain \( \mathbb{Q}(\zeta) \), where \( \zeta \) is a primitive 3rd root of unity.

5. Let \( \theta \) be a root of \( f(x) = x^3 - 3x + 1 \). Show that \( K = \mathbb{Q}(\theta) \) is a splitting field for \( f \) and that \( \text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z} \). Express the other two roots of \( f \) explicitly in the form \( a + b\theta + c\theta^2 \) for \( a, b, c \in \mathbb{Q} \).

6. Section 14.6 #19.