1. Fix a prime \( p \). Show that the following subgroup of \( \text{GL}_2 \mathbb{F}_p \) is solvable:

\[
B = \left\{ \left( \begin{array}{cc} x & z \\ 0 & y \end{array} \right) \right| x, y \in \mathbb{F}_p^\times, z \in \mathbb{F}_p \right\}
\]

Here, the group operation is just matrix multiplication.

2. (a) Prove directly from the definition that \( S_4 \) is solvable.

(b) Prove that \( A_5 \) is simple using the following outline.

- (i) Show \( A_5 \) has 5 distinct conjugacy classes of elements, and count the number of elements in each class.
- (ii) For any normal subgroup \( H \triangleleft G \) show that \( H \) is a union of conjugacy classes of \( G \).
- (iii) If \( N \triangleleft A_5 \) use that \( |N| \) divides \( |A_5| \) and parts (i) and (ii) to show that \( N = \{1\} \) or \( A_5 \).

Alternatively, give a geometric proof using the fact that \( A_5 \) is the group of Euclidean isometries of a regular dodecahedron.

Remark: \( A_5 \) is the smallest of all the simple groups. In fact, every group of order less than 60 is solvable.

(c) Use (b) to show that \( S_n \) is not solvable for \( n \geq 5 \).

3. (Section 14.7, #12) Let \( L \) be the Galois closure of a finite extension \( \mathbb{Q}(\alpha) \) over \( \mathbb{Q} \). If \( p \) is a prime dividing the order of \( \text{Gal}(L/\mathbb{Q}) \), show that there is a subfield \( F \) of \( L \) with \( [L:F] = p \) and \( L = F(\alpha) \).

Hint: You’ll need to use Theorem 18 from Section 4.5: if \( p \) is a prime dividing the order of a finite group \( G \), then \( G \) has an element of order \( p \).

4. (Section 14.7, #13) Let \( F \subset \mathbb{R} \) be a field. Let \( a \) be an element of \( F \) which has a real \( n^{th} \) root \( \alpha = \sqrt[n]{a} \), and set \( K = F(\alpha) \). Prove that if \( L \) is any Galois extension of \( F \) contained in \( K \) then \( [L:F] \leq 2 \).

5. For a field \( k \), here are some basic problems for varieties in \( k^2 \), where we take the coordinates to be \((x, y)\). Except for part (b), assume that \( k \) is algebraically closed.

(a) Let \( V \) be the \( x \)-axis, i.e. \( V = V(y) \). Prove that \( V \) is irreducible. Hint: Show a prime ideal is radical.

(b) Give an example of a field \( k \), necessarily not algebraically closed, for which the \( x \)-axis is reducible.

(c) Prove that \( V = V(x - y) \) is irreducible.

(d) Prove that \( S = \{(a, a) \in k^2 \mid a \neq 1\} \) is not an algebraic variety if \( k = \mathbb{C} \).

(e) What is the decomposition of \( V = V(x^2 - y^2) \) into irreducibles? Warning: The answer depends on \( k \)!