1. Fix a prime $p$. Show that the following subgroup of $GL_2(F_p)$ is solvable:

$$B = \left\{ \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \middle| x, y \in F_p^\times, z \in F_p \right\}$$

Here, the group operation is just matrix multiplication.

2. (a) Prove directly from the definition that $S_4$ is solvable.

(b) Prove that $A_5$ is simple using the following outline.

(i) Show $A_5$ has 5 distinct conjugacy classes of elements, and count the number of elements in each class.

(ii) For any normal subgroup $H \triangleleft G$ show that $H$ is a union of conjugacy classes of $G$.

(iii) If $N \triangleleft A_5$ use that $|N|$ divides $|A_5|$ and parts (i) and (ii) to show that $N = \{1\}$ or $A_5$.

Alternatively, give a geometric proof using the fact that $A_5$ is the group of Euclidean isometries of a regular dodecahedron.

Remark: $A_5$ is the smallest of all the simple groups. In fact, every group of order less than 60 is solvable.

(c) Use (b) to show that $S_n$ is not solvable for $n \geq 5$.

3. (Section 14.7, #12) Let $L$ be the Galois closure of a finite extension $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. If $p$ is a prime dividing the order of $Gal(L/\mathbb{Q})$, show that there is a subfield $F$ of $L$ with $[L:F] = p$ and $L = F(\alpha)$.

Hint: You'll need to use Theorem 18 from Section 4.5: if $p$ is a prime dividing the order of a finite group $G$, then $G$ has an element of order $p$.

4. (Section 14.7, #13) Let $F \subset \mathbb{R}$ be a field. Let $a$ be an element of $F$ which has a real $n^{th}$ root $\alpha = \sqrt[n]{a}$, and set $K = F(\alpha)$. Prove that if $L$ is any Galois extension of $F$ contained in $K$ then $[L:F] \leq 2$.

5. For a field $k$, here are some basic problems for varieties in $k^2$, where we take the coordinates to be $(x, y)$. Except for part (b), assume that $k$ is algebraically closed.

(a) Let $V$ be the $x$-axis, i.e. $V = V(y)$. Prove that $V$ is irreducible. Hint: Show a prime ideal is radical.

(b) Give an example of a field $k$, necessarily not algebraically closed, for which the $x$-axis is reducible.

(c) Prove that $V = V(x - y)$ is irreducible.

(d) Prove that $S = \{(a, a) \in k^2 \mid a \neq 1\}$ is not an algebraic variety if $k = \mathbb{C}$.

(e) What is the decomposition of $V = V(x^2 - y^2)$ into irreducibles? **Warning:** The answer depends on $k$!