1. Consider the parabola \( y = x^2 \) in \( \mathbb{R}^2 \).

   (a) Find a polynomial \( f \in \mathbb{R}[x, y, z] \) so that the variety \( V \) in \( \mathbb{P}^2_\mathbb{R} \) defined by \( f \) has \( V \cap \mathbb{R}^2 \) the above parabola. Here \( \mathbb{R}^2 = \{(x : y : 1)\} \).

   (b) How many points does \( V \) have in the line at infinity \( \mathbb{P}^1_\infty = \{(x : y : 0)\} \)?

   (c) Using a projective transformation, show that \( V \) is in fact tangent to \( \mathbb{P}^1_\infty \).

   (d) Find a projective transformation so that \( p_A(V) \cap \mathbb{R}^2 \) is a hyperbola.

2. Let \( k \) be a field. A line in \( \mathbb{P}^2_k \) is the variety corresponding the equation \( ax + by + cz = 0 \), where \( a, b, c \in k \) are not all zero.

   (a) Show that, up to change of coordinates, all lines are the same. That is, given two lines \( L, L' \) there exists a matrix \( A \in \text{GL}_3(k) \) so that the corresponding projective transformation \( p_A: \mathbb{P}^2_k \rightarrow \mathbb{P}^2_k \) takes \( L \) to \( L' \).

   (b) Prove that any two distinct points \( p_1, p_2 \in \mathbb{P}^2_k \) determine a unique line.

   (c) Prove that any two distinct lines intersect in exactly one point.

   **Hint:** What object in \( k^3 \) corresponds to a line in \( \mathbb{P}^2_k \)?

3. Let \( k \) be a field, and consider \( \mathbb{P} = \mathbb{P}^2_k \).

   (a) Let \( p_1, p_2, p_3, p_4 \) be points in \( \mathbb{P} \) so that no three are colinear, i.e. no three lie on a common line. Show there is a projective change of coordinates so that the \( p_i \) become \( (1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1) \).

   (b) Find all conics passing through the five points

   \[
   (1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1), (a : b : c)
   \]

   (c) Suppose \( p_1, \ldots, p_5 \) are points in \( \mathbb{P} \) with no four colinear. Use (a-b) to show there is at most one conic containing all 5 points.

   **Note:** This is one of many illustrations of the power of changing coordinates.

4. Consider the plane curve \( X = \mathbf{V}(x^3 y + y^3 z + z^3 x) \) in \( \mathbb{P}_\mathbb{C}^2 \).

   (a) Find \( X \cap L_\infty \), where \( L_\infty \) is the line at infinity, i.e. \( \mathbf{V}(z) \).

   (b) Prove that \( X \) is smooth, being sure to include those points found in (a).

   **Note:** Any smooth curve in \( \mathbb{P}^2_\mathbb{C} \) is automatically irreducible, and has genus \( \binom{d-1}{2} \), where \( d \) is the degree of the defining polynomial. Hence, as topological space, \( X \) is as shown below.
(c) \( X \) is very symmetric. Find a group of projective transformations of order 21 that leaves \( X \) invariant. In fact, the full group of such projective automorphisms has order 168 and is the simple group \( \text{PSL}_2 \mathbb{F}_7 \). In fact, this is the most symmetries that a genus 3 curve can have...

5. In this problem, you'll explore elliptic curves in \( \mathbb{P}_\mathbb{R}^2 \). In addition to the points in \( \mathbb{R}^2 \) given by a standard equation \( y^2 = x(x^2 + ax + b) \), there is an additional point at infinity which is the identity element in the group law. Note: Elliptic curves are always taken to be smooth, as otherwise the group law gets confusing.

One thing that wasn't mentioned in class is how to add a point \( p \) to itself. In this case, one takes the tangent line at \( p \) as shown:

(a) Consider the curve \( E \) given by \( y^2 = x^3 + 4x \). Show that \((2, 4)\) has order 4.

(b) Now consider an arbitrary elliptic curve \( E \). Explain why any point in \( E \) of the form \((x, 0)\) has order 2 in \( E \).

(c) Find the subgroup of \( E \) consisting of all points of order 2 (plus the identity element), and identify it as a group. Note: there are two cases here, depending on the specific curve \( E \).