Lecture 35: More on Projective Space

Last time: $\mathbb{P}^n_k = \left\{ \begin{array}{l} \text{all lines in} \\ k^{n+1} \text{ through } 0 \end{array} \right\} = \{x \in k^{n+1} \mid x \neq 0\} \Big/ \begin{array}{l} x \sim \lambda x \\ \text{for } \lambda \in k^\times \end{array}$

$$= \{(x_1 : x_2 : \cdots : x_n : 1)\} \cup \{(\underbrace{x_1 : x_2 : \cdots : x_n}_{\text{not all } 0} : 0)\}$$

$$= \mathbb{A}^n_k \cup \mathbb{P}^{n-1}_k .$$ [Example of a Moduli Space]

Continue to focus on $\mathbb{P}^2_{\mathbb{R}} =$  $\Big/ x \sim -x$

Recall $f \in \mathbb{R}[x,y,z]$ is homogenous if each term $c\,x^i y^j z^k$ has the same total degree $i+j+k$.

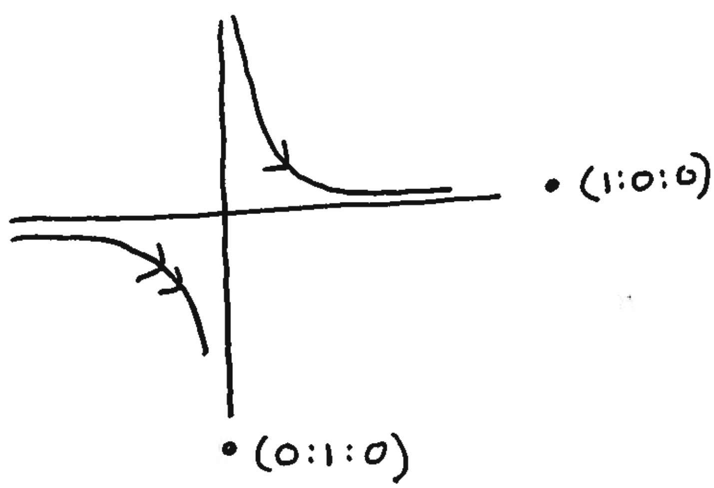
Such an f gives a projective variety:

$$\mathbb{V}(f) = \{(a:b:c) \in \mathbb{P}^2_{\mathbb{R}} \mid f(a,b,c) = 0\}$$

Ex: $V = \mathbb{V}(xy - z^2)$

$V \cap \mathbb{R}^2 = \mathbb{V}_{\mathbb{R}^2}(xy-1)$

$V \cap \mathbb{P}^1_{\mathbb{R}} = \{(1:0:0), (0:1:0)\}$



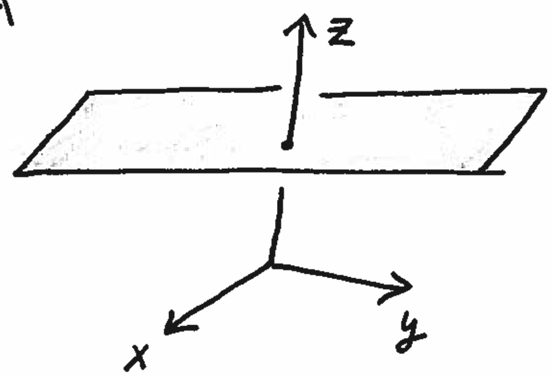
Goal: Show V is the "same" as $V_{\mathbb{R}^2}(x^2+y^2-1)$.

Symmetries of $\mathbb{P}^2_{\mathbb{R}}$: Let $A \in GL_3\mathbb{R}$, i.e. a 3×3 matrix with $\det(A) \neq 0$. Gives a projective transformation

$$P_A : \mathbb{P}^2_{\mathbb{R}} \longrightarrow \mathbb{P}^2_{\mathbb{R}} \quad \text{via } x \longmapsto Ax \text{ for } x \neq 0 \text{ in } \mathbb{R}^3,$$

which makes sense as $\lambda x \longmapsto A(\lambda x) = \lambda(Ax)$.

Ex: If $A = \begin{pmatrix} B & \begin{smallmatrix}0\\0\end{smallmatrix} \\ \begin{smallmatrix}0 & 0\end{smallmatrix} & 1 \end{pmatrix}$, then P_A sends $\mathbb{R}^2 = \{(x:y:1)\}$ to itself and acts on it via the linear trans. with matrix B.



Ex: $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ gives

$$P_A(x:y:1) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+2 \\ y+3 \\ 1 \end{pmatrix} = (x+2 : y+3 : 1)$$

Which acts on $\mathbb{R}^2 \subseteq \mathbb{P}^2_{\mathbb{R}}$ as a translation. What happens to $\mathbb{P}^1_{\mathbb{R}} = \{(x:y:0)\}$? There P_A is the identity!

Ex: $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\quad P_A(x:y:z) = (z:x:y)$

Takes $\mathbb{P}^1_{\mathbb{R}} = \{(x:y:0)\}$ to $\{(0:x:y)\} = \underbrace{\{(0:x:1)\}}_{y\text{-axis in } \mathbb{R}^2.} \cup \{(0:1:0)\}$

Moral: Nothing special about this particular $\mathbb{P}^1_{\mathbb{R}}$ at infinity, it's just another line!

Cor: $\mathbb{P}^2_{\mathbb{R}} \setminus \{y\text{-axis} \cup \{0:1:0\}\}$ is also a copy of $\mathbb{R}^2$!

Ex: $A = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix}$ $\qquad V = \mathbb{V}(xy - z^2) \subseteq \mathbb{P}^2_{\mathbb{R}}$

Q: What is $P_A(V)$? [Should be another variety.]

Consider

$$\begin{tikzcd} \mathbb{P}^2_{\mathbb{R}} \arrow[r, shift right, "P_{A^{-1}}"'] & \mathbb{P}^2_{\mathbb{R}} \arrow[l, shift right, "P_A"'] \arrow[r, "f"] & \mathbb{R} \\ P_A(V) \arrow[u, phantom, "\cup"] & V = \{f=0\} \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

Key: $P_A(V) = \{ f \circ P_{A^{-1}} = 0 \} = \mathbb{V}(f \circ P_{A^{-1}})$

Let's use coor: $\mathbb{P}^2_{\mathbb{R}} \xrightarrow{} \mathbb{P}^2_{\mathbb{R}}$

$(u:v:w) \quad p_A^{-1} \quad (x:y:z)$

So

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v+w \\ -v+w \\ u \end{pmatrix}$$

and hence

$$f \circ p_A^{-1}(u,v,w) = f(v+w, -v+w, u) = (v+w)(-v+w) - u^2$$
$$= -v^2 + w^2 - u^2$$

Thus

$$V' = p_A(V) = \mathbb{V}_{\mathbb{P}^2_{\mathbb{R}}}(u^2+v^2-w^2) = \mathbb{V}_{\mathbb{R}^2}(u^2+v^2-1)$$

no points at ∞!

So in $\mathbb{P}^2_{\mathbb{R}}$ have

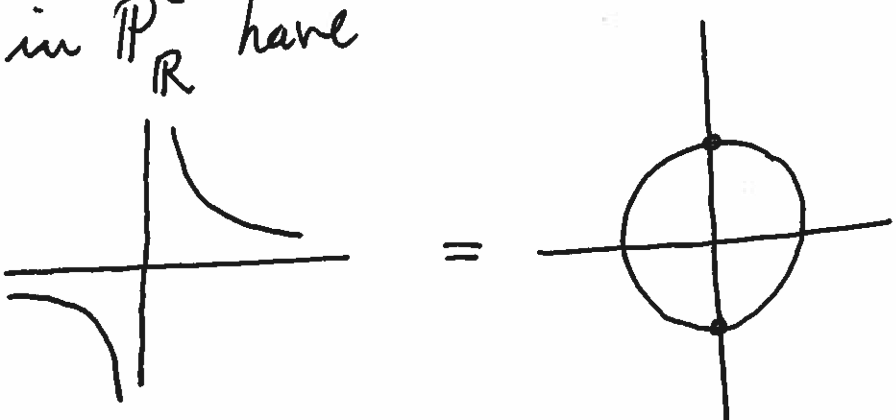


image of $\{(x:y:0)\} = \mathbb{P}^1_\infty$.

<u>General Conic</u>: In $\mathbb{R}^2$, consider the variety of

$$q(x,y) = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f$$

which comes from $\swarrow$ <u>homogenousization</u>

$$V = \mathbb{V}_{\mathbb{P}^2_{\mathbb{R}}}\left(ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2 \right)$$

Have

$$q(x,y,z) = (x\ y\ z) \overbrace{\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}}^{M} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

which is a quadratic form on $\mathbb{R}^3$. $\exists A \in GL_3\mathbb{R}$ such that $A^T M A = \begin{pmatrix} \varepsilon_1 & & 0 \\ & \varepsilon_2 & \\ 0 & & \varepsilon_3 \end{pmatrix}$ with $\varepsilon_i \in \{-1, 0, 1\}$.

[Why? If nec. discuss at length]

So $V' = P_{A^{-1}}(V)$ is given by

$$q'(u,v,w) = \left(A \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right)^T M \left(A \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right) = (u\,v\,w) \begin{pmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$= \varepsilon_1 u^2 + \varepsilon_2 v^2 + \varepsilon_3 w^2$$

<u>Thm</u>: Up to projective transformations, any conic in $\mathbb{P}^2_{\mathbb{R}}$ is one of

(a) $\mathbb{V}(x^2+y^2-z^2)$ (non-deg. conic)

(b) $\mathbb{V}(x^2+y^2+z^2) = \emptyset$

(c) $\mathbb{V}(x^2-y^2)$ = two lines

(d) $\mathbb{V}(x^2+y^2) = \{(0:0:1)\}$

(e) $\mathbb{V}(x^2)$ = y-axis

(f) $\mathbb{V}(0) = \mathbb{P}^2_{\mathbb{R}}$

Let C = nondegen. conic in $\mathbb{P}^2_{\mathbb{R}}$

L = line at ∞ = $\{(x:y:0)\}$

Three cases:

$C \cap L = \emptyset \implies C \cap \mathbb{R}^2$ is an ellipse

$C \cap L = 2$ pts $\implies C \cap \mathbb{R}^2$ is a hyperbola

$C \cap L = 1$ pt, $\implies C \cap \mathbb{R}^2$ is a parabola.

tangent intersection

$\mathbb{V}_{\mathbb{R}^2}(x^2-y)$ = 

$\subseteq \mathbb{V}_{\mathbb{P}^2_{\mathbb{R}}}(x^2-yz)$

$$\Big\downarrow P\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

$\mathbb{V}_{\mathbb{P}^2_{\mathbb{R}}}(v^2-uw)$