Practical solutions to hard problems in 3-dimensional topology.

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Fields Institute, November 20, 2009

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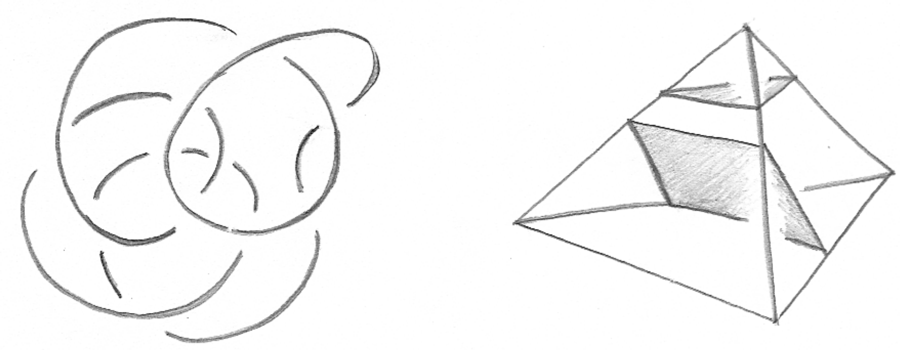
[Haken 1961] Whether a knot in S^3 is unknotted. More generally, find the simplest surface representing a class in $H_2(M;\mathbb{Z})$.



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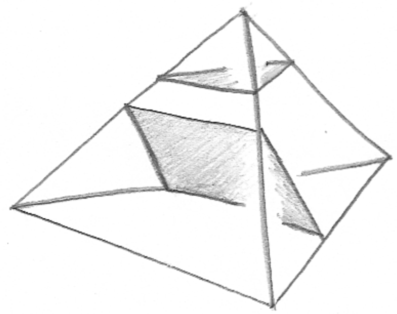

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Thurston and Perelman: 3-manifolds have canonical decompositions into geometric pieces modeled on $\mathbb{E}^3$, S^3, $\mathbb{H}^3$, $S^2\times\mathbb{R}$, $\mathbb{H}^2\times\mathbb{R}$, Nil, Sol, $\widetilde{\mathrm{SL}_2\mathbb{R}}$.

The work of Perelman, Casson-Manning, Epstein et. al., Hodgson-Weeks, Jaco-Oertel, Haken-Hemion-Matveev, Casson, Rubinstein-Thompson, and others gives

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How hard are these questions?

[Agol-Hass-Thurston 2002] The following is NP-complete:

Q: Given a manifold M, a knot K in $\mathcal{T}^1$, and $g \in \mathbb{N}$, is there a surface $\Sigma \subset M$ with boundary K and genus $\leq g$?

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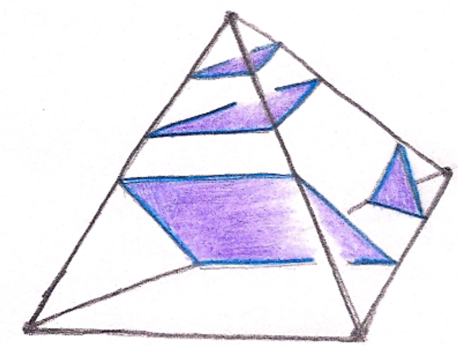
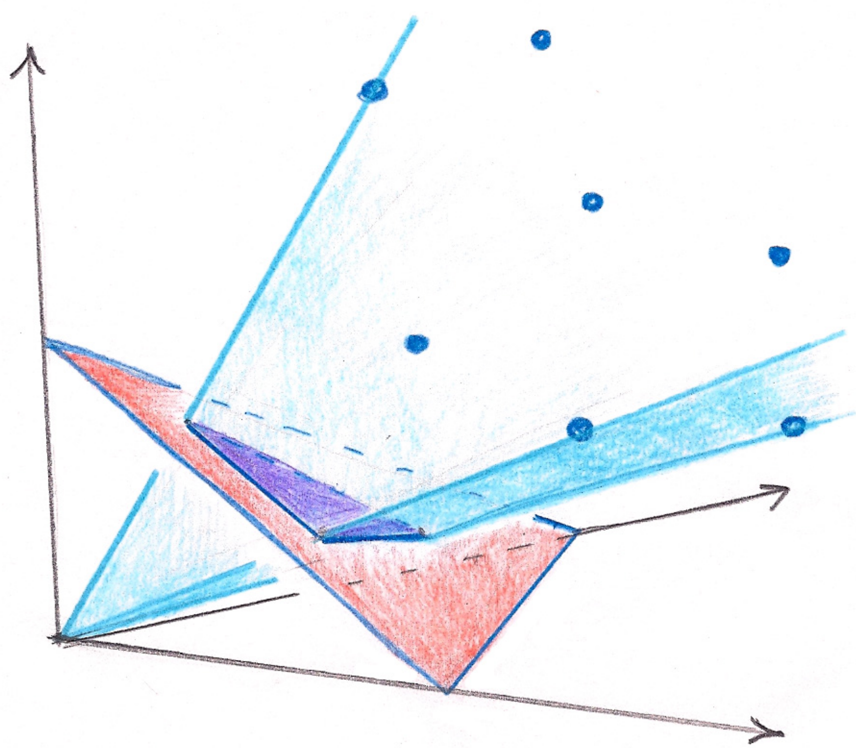
Normal surfaces meet each tetrahedra in a standard way:



and correspond to certain lattice points in a finite polyhedral cone in $\mathbb{R}^{7t}$ where $t = \#\mathcal{T}$:



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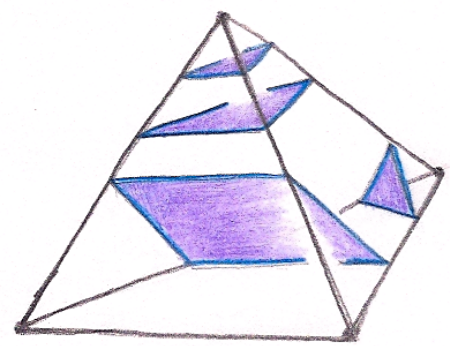

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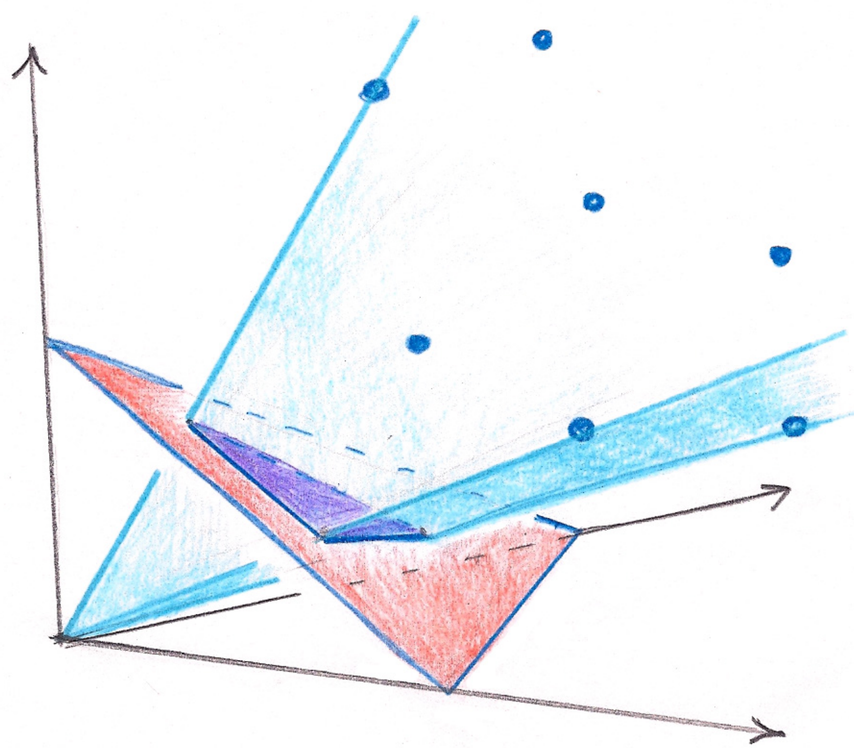

Meta Thm. *In an interesting class of surfaces, there is one which is normal. Moreover, one lies on a vertex ray of the cone.*

E.g. The class of minimal genus surfaces whose boundary is a given knot.

Problem: the dimension grows linearly with t, and moreover there can be exponentially many vertex rays. In practice, limited to $t < 40$.

Worse, sometimes have a second step examining each $M \setminus \Sigma$ and looking for surfaces there, and that new manifold may be much more complicated than M itself.

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Thm. (Dunfield-Ramakrishnan 2007) *There is a closed hyperbolic 3-manifold M of arithmetic type, with an infinite family of finite covers $\{M_n\}$ of degree d_n, where the number ν_n of fibered faces of the Thurston norm ball of M_n satisfies*

$$\nu_n \geq \exp\left(0.3\frac{\log d_n}{\log\log d_n}\right) \quad \textit{as } d_n \to \infty.$$

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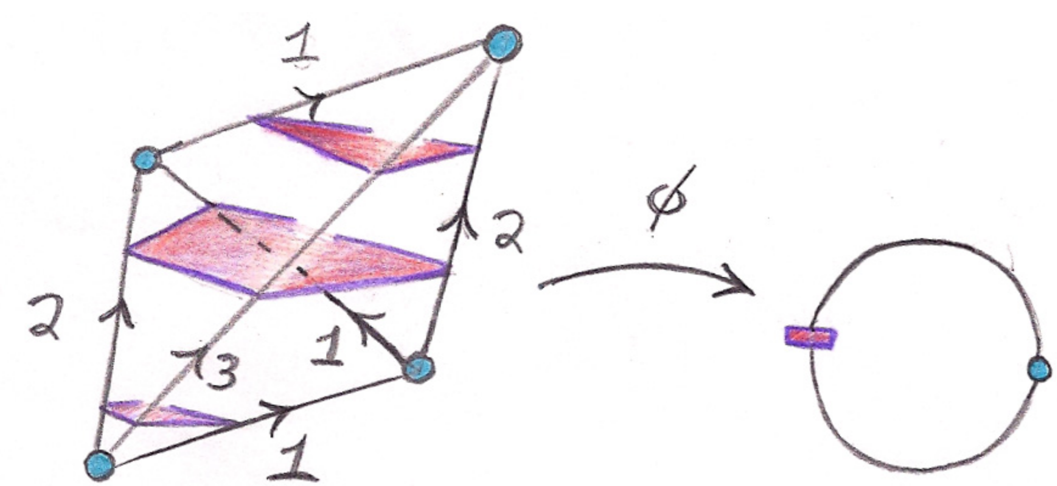
Practical Trick 1: Finding the simplest surface representing some $\phi \in H^1(M;\mathbb{Z}) \cong H_2(M;\mathbb{Z})$.

Use a triangulation with only one vertex (cf. Casson, Jaco-Rubinstein). The ϕ comes from a unique 1-cocycle, which realizes ϕ as a piecewise affine map $M \to S^1$.



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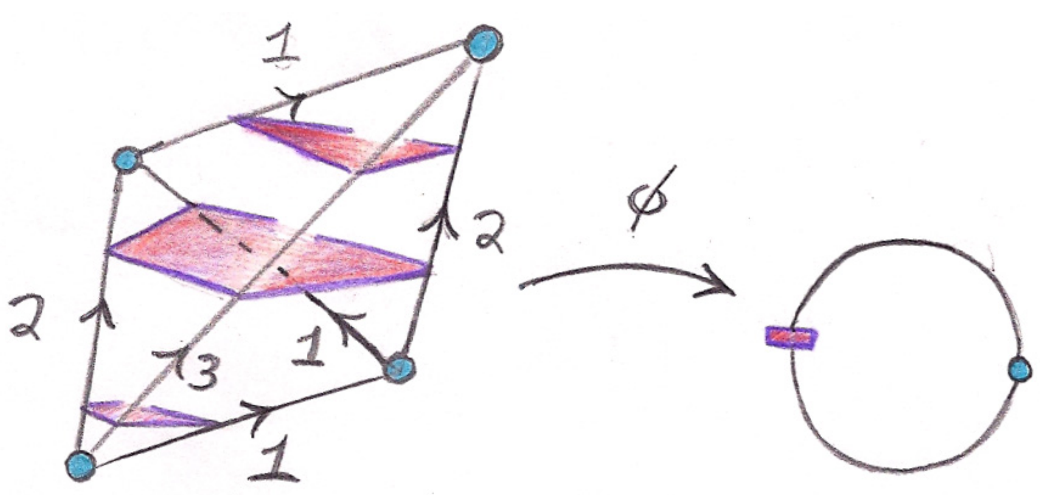

Power of randomization: Trying several different triangulations usually yields the minimal genus surface.

Lower bounds on the genus come from (twisted) Alexander polynomials.

Practical Trick 2: Proving that $N = M \setminus \Sigma$ is $\Sigma \times I$.

Start with a presentation for $\pi_1(N)$ coming from a triangulation, and then simplify that it using Tietze transformations. With luck (i.e. randomization), one gets a one-relator presentation of a surface group. This gives $N \cong \Sigma \times I$ by [Stallings 1960].

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Rank vs. genus (with Helen Wong)

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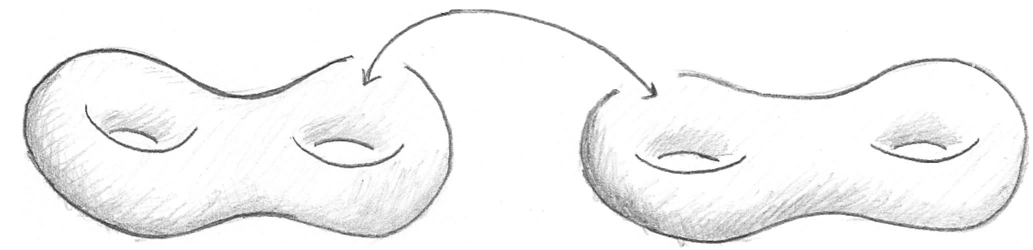

Consider

$$\operatorname{rank}(M) = \text{min genus of a Heegaard splitting}$$
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Clearly have $\operatorname{rank}(M) \leq \operatorname{genus}(M)$.

Q. *Does* $\operatorname{rank}(M) = \operatorname{genus}(M)$ *for all hyperbolic 3-manifolds?*

[Boileau-Zieschang 1984] There are Seifert fibered spaces with $\operatorname{rank}(M) \neq \operatorname{genus}(M)$.

Further graph manifold examples found by Weidmann and Schultens.

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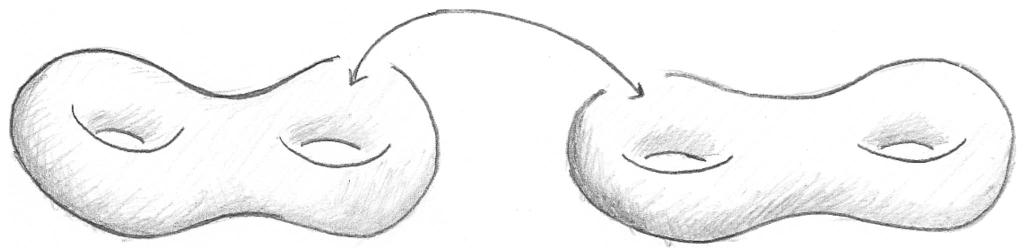

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Searching for an example.

Computability in theory for M^3 hyperbolic:

- **Rank:** Yes [Kapovich-Weidmann 2004]
- **Genus:** Unknown, likely yes. Rubinstein and Stocking showed that (many) Heegaard surfaces can be made almost normal, but there are infinitely many candidates surfaces.

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Computability in practice.

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- **Genus:** Sometimes. Start with a presentation of $\pi_1(M)$ coming from a triangulation, then simplify via Tietze transformations. The result inevitably comes from a Heegaard splitting of M. Using randomization, can get a good idea of what the genus should be. Lower bounds, other than the rank, are few, e.g. quantum invariants.

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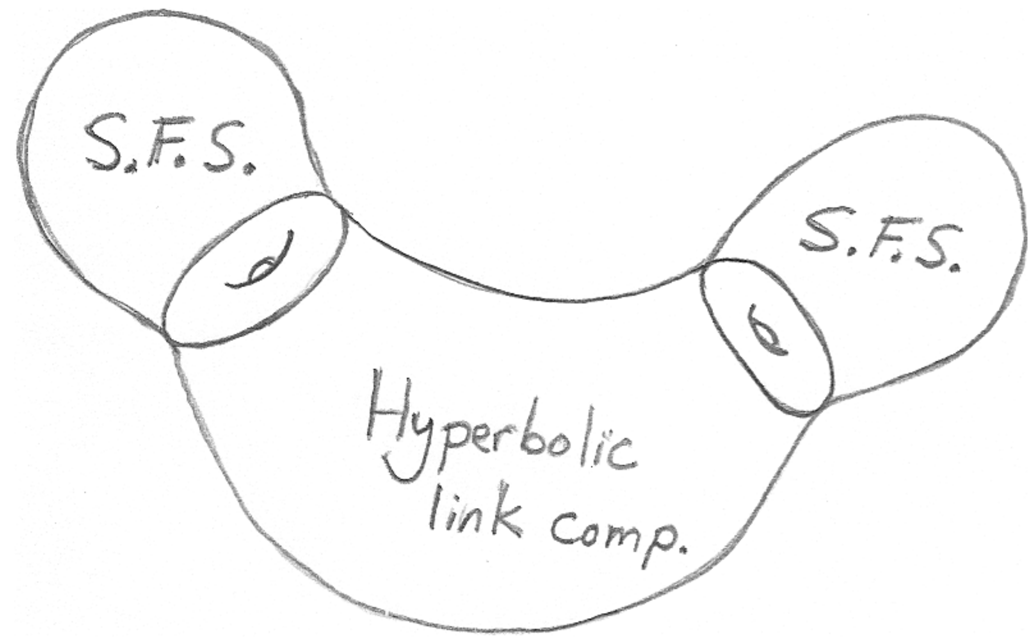

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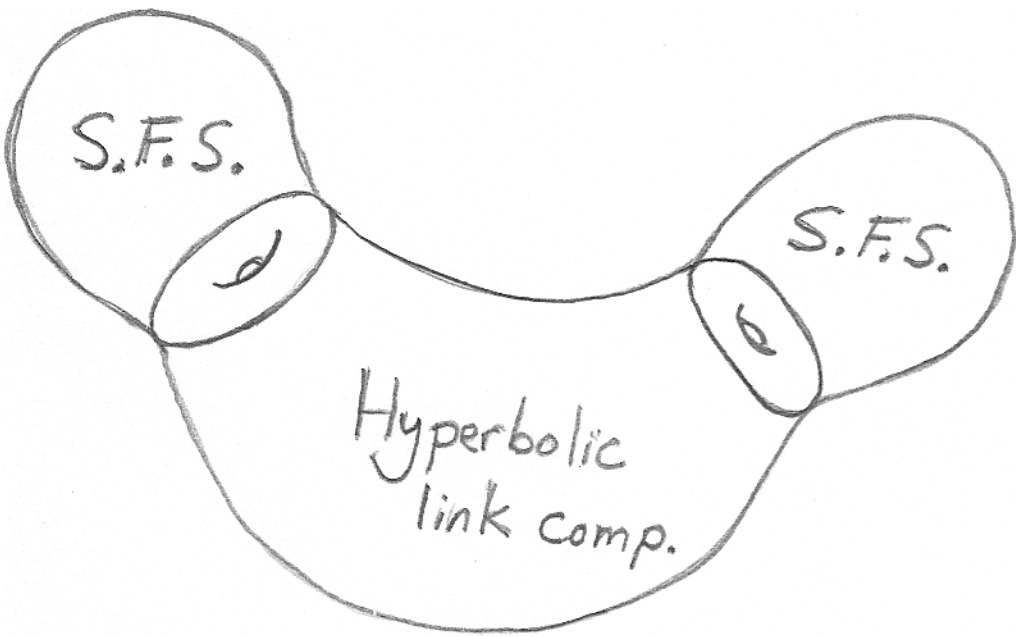

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SnapPy

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SnapPy is a user interface to the SnapPea kernel which runs on Mac OS X, Linux, and Windows. SnapPy combines a link editor and 3D-graphics for Dirichlet domains and cusp neighborhoods with a powerful command-line interface based on the Python programming language. You can see it in action, learn how to install it, and read the tutorial.

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- Screenshots: SnapPy in action
- Installing and running SnapPy
- Tutorial
- snappy: A Python interface for SnapPea
- Using SnapPy's link editor
- To Do List
- Development Basics: OS X
- Development Basics: Windows XP

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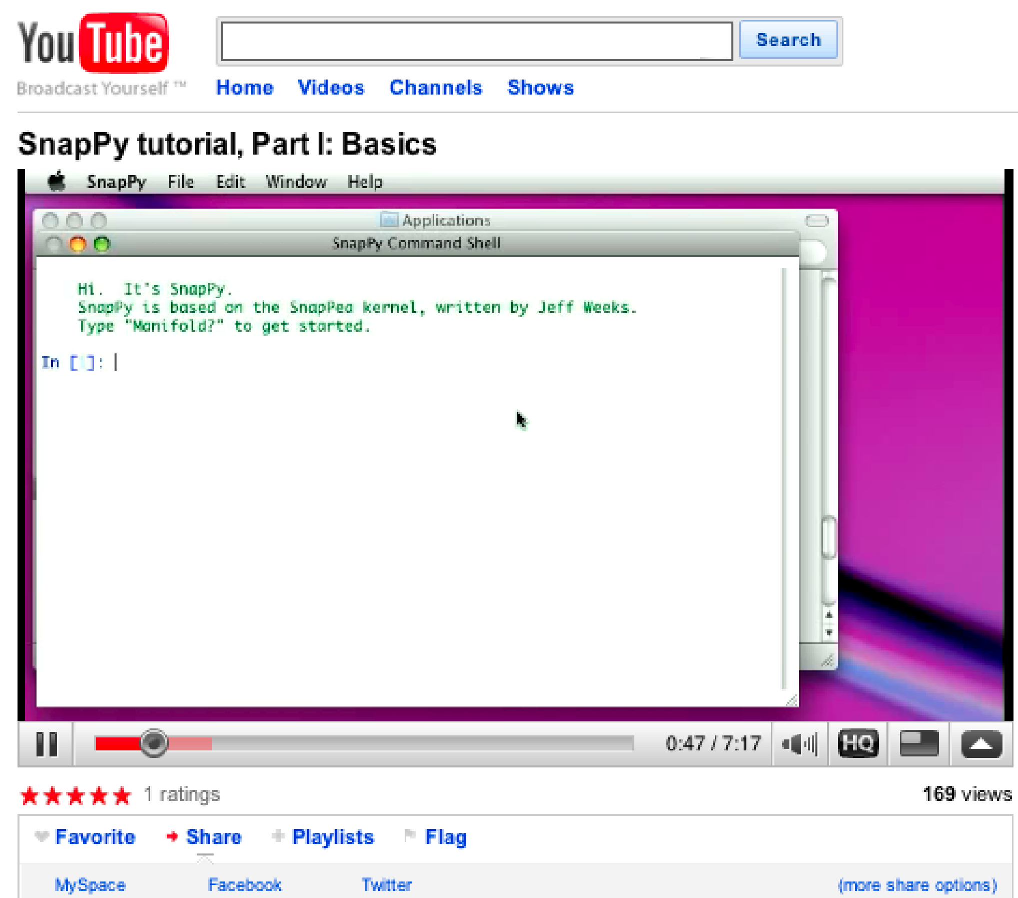
http://www.youtube.com/user/NathanDunfield/



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