Floer homology, orderable groups, and taut foliations of hyperbolic 3-manifolds:

An experimental study

Nathan M. Dunfield
(University of Illinois)

These slides already posted at:

http://dunfield.info/slides/Newt17.pdf

Y^3: closed oriented irreducible with $H_*(Y;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$.

Conj: For an irreducible $\mathbb{Q}$HS Y, TFAE:

(a) $\widehat{HF}(Y)$ is non-minimal.

(b) $\pi_1(Y)$ is left-orderable.

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Heegaard Floer Homology: An $\mathbb{F}_2$-vector space $\widehat{HF}(Y)$, part of a $3+1$ dimensional (almost) TQFT.

[Kronheimer, Mrowka, Ozsváth, Szabó 2003] No Dehn surgery on a nontrivial knot in S^3 yields $\mathbb{R}\mathrm{P}^3$.

Basic fact: $\dim \widehat{HF}(Y) \geq \left|H_1(Y;\mathbb{Z})\right|$. When equal, Y is an *L-space*.

L-spaces: Spherical manifolds, e.g. $L(p,q)$.

Non-L-spaces: $1/n$-Dehn surgery on a knot in S^3 other than the unknot or the trefoil.

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Left-order: A total order on a group G where $g < h$ implies $f \cdot g < f \cdot h$ for all $f, g, h \in G$.

Orderable: $(\mathbb{R}, +)$, $(\mathbb{Z}, +)$, F_n, B_n.

Non-orderable: finite groups, $\mathrm{SL}_n \mathbb{Z}$ for $n \geq 3$.

For countable G, equivalent to $G \hookrightarrow \mathrm{Homeo}^+(\mathbb{R})$.

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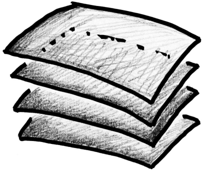

Taut foliation: A decomposition $\mathscr{F}$ of Y into 2-dim'l leaves where:

(a) Smoothness: $C^{1,0}$

(b) Co-orientable.

(c) There exists a loop transverse to $\mathscr{F}$ meeting every leaf.

Example: Y fibers over S^1.

Better example: T^3 foliated by irrational planes.

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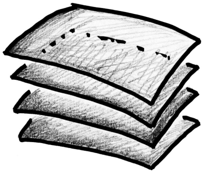
Non-examples:

While every closed 3-manifold has a foliation $\mathscr{F}$ satisfying (a) and (b), if $\mathscr{F}$ is taut then $\widetilde{Y}$ is $\mathbb{R}^3$ or $S^2 \times \mathbb{R}$ and so $\pi_1(Y)$ is infinite.

The hyperbolic 3-manifold of least volume, the Weeks manifold, is a $\mathbb{Q}$HS which has no taut foliations.



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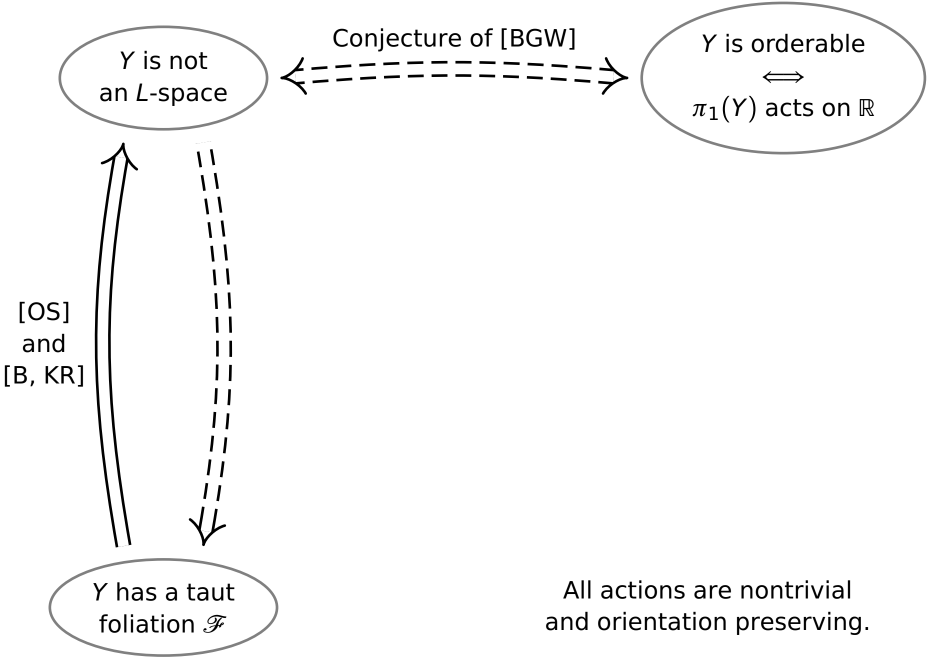
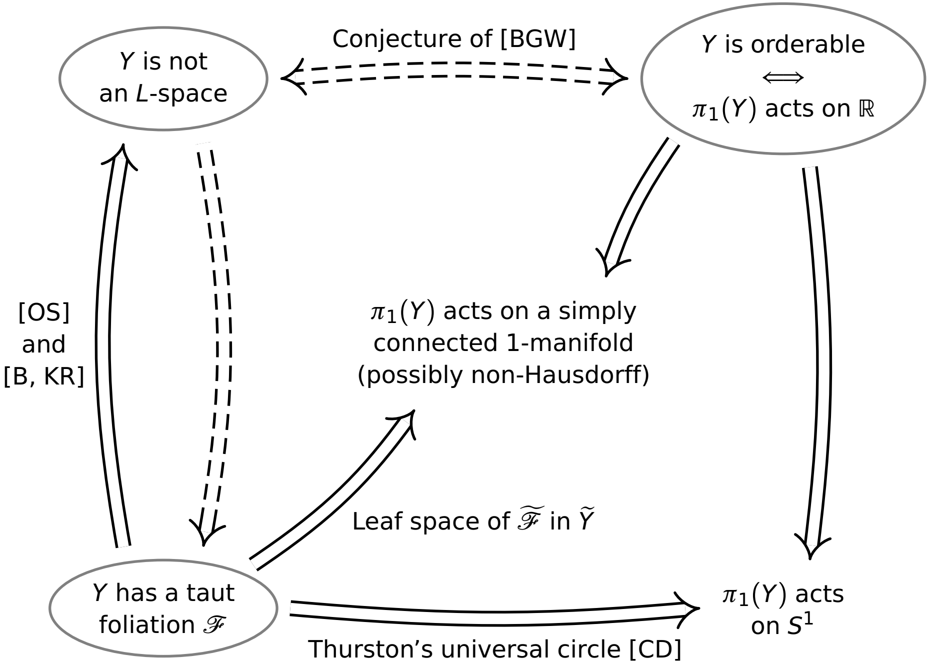


Conjecture of [BGW]
Y is not an L-space
Y is orderable $\Longleftrightarrow$ $\pi_1(Y)$ acts on $\mathbb{R}$
[OS] and [B, KR]
Y has a taut foliation $\mathscr{F}$
All actions are nontrivial and orientation preserving.



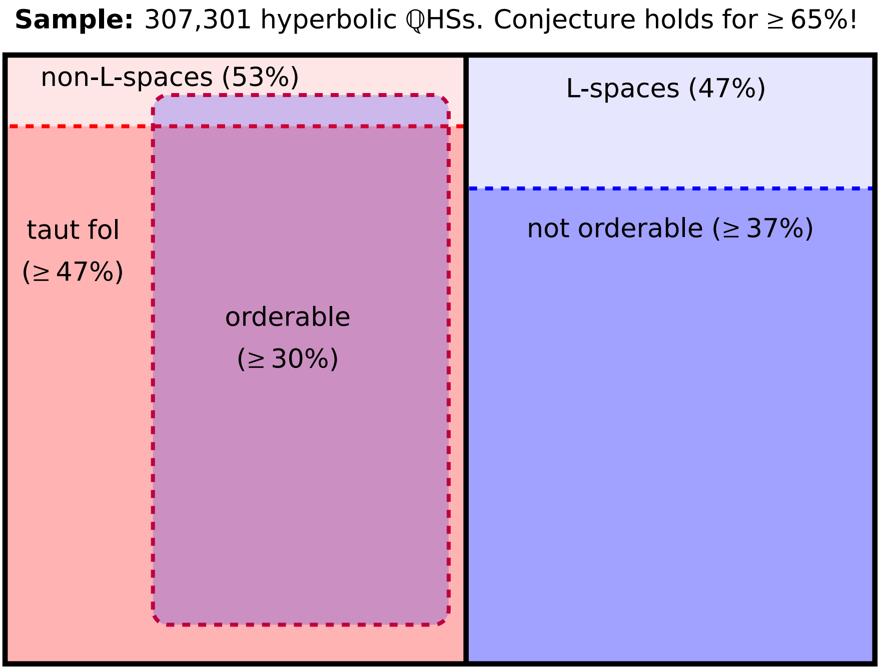
Conjecture of [BGW]
Y is not an L-space
Y is orderable $\iff$ $\pi_1(Y)$ acts on $\mathbb{R}$
$\pi_1(Y)$ acts on a simply connected 1-manifold (possibly non-Hausdorff)
[OS] and [B, KR]
Leaf space of $\widetilde{\mathscr{F}}$ in $\widetilde{Y}$
Y has a taut foliation $\mathscr{F}$
$\pi_1(Y)$ acts on S^1
Thurston's universal circle [CD]

Evidence for the conjecture:

[Hanselman-Rasmussen^2-Watson + Boyer-Clay 2015] True for all graph manifolds.

[Culler-D. 2016 + Roberts 2001] Suppose $K \subset S^3$ where $\Delta_K(t)$ has a simple root on the unit circle and which is lean. Then there exists $\epsilon > 0$ so that the conjecture holds for the r Dehn surgery on K whenever $r \in (-\epsilon, \epsilon)$.

[Gordon-Lidman, Tran, . . .]


Sample: 307,301 hyperbolic ℚHSs. Conjecture holds for ≥ 65%!
non-L-spaces (53%)
L-spaces (47%)
taut fol
(≥ 47%)
orderable
(≥ 30%)
not orderable (≥ 37%)

Starting point:

$$\mathscr{C} = \left\{ \begin{array}{c} \text{hyp } \mathbb{Q}\text{-homology solid tori} \\ \text{triang by} \leq 9 \text{ ideal tets} \\ \text{[Burton 2014]} \end{array} \right\}$$

$$\mathscr{Y} = \left\{ \begin{array}{c} \text{hyp } \mathbb{Q}\text{HS fillings on } C \in \mathscr{C} \\ \text{with systole} \geq 0.2 \end{array} \right\}$$

$$\#\mathscr{C} = 59{,}068 \qquad \#\mathscr{Y} = 307{,}301$$

Mean vol$(Y \in \mathscr{Y})$ is 6.9 with $\sigma = 0.9$.

59% of $Y \in \mathscr{Y}$ have a unique Dehn filling description involving $\mathscr{C}$; the remaining 41% average 3.4.

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Determining L-spaces

Alg. decidable [Sarkar-Wang 2006]

Bordered Floer [LOT, L-Zhan]

A $\mathbb{Q}$-homology solid torus M is **Floer simple** if it has at least two L-space Dehn fillings.

[Rasmussen² 2015] If you know two L-space fillings on M, then the precise set of L-space fillings can be read off from the Turaev torsion of M.

[Berge; D 2015] There are at least 54,790 finite fillings on $C \in \mathscr{C}$.

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L-sp	non-L	L-sp?	F-simp	non-F	simp?	
0	0	100%	0	0	100%	init state

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47%	53%	0%	51%	14%	35%	foliations + crank

(*) Here 0% is really 518 manifolds, or 0.17%.

Finding 143,516 taut folations.

$\mathscr{T}$ a 1-vertex triangulation of Y.

Def. A *laminar orientation* of $\mathscr{T}$ is:

(a) An orientation of the edges where every face is acyclic.

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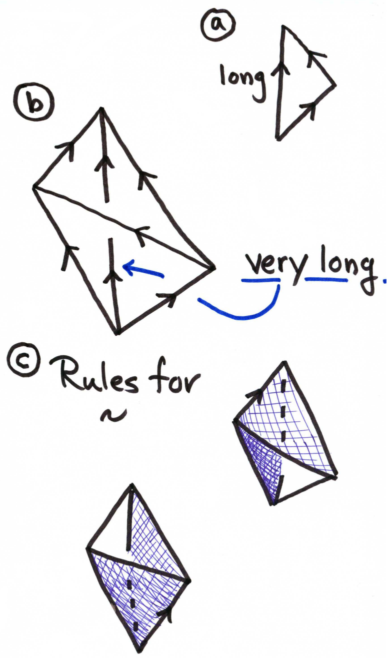
[D. 2015] *If Y has a tri with a laminar orient, then Y has a taut foliation.*



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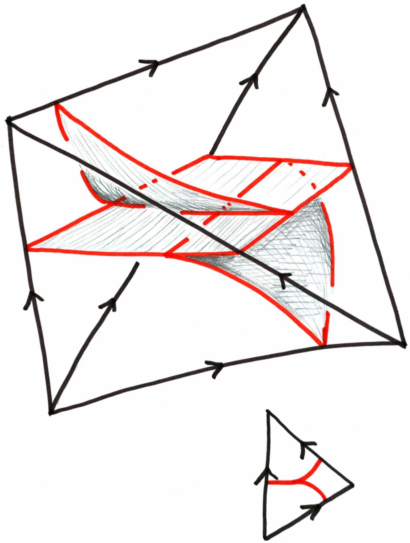

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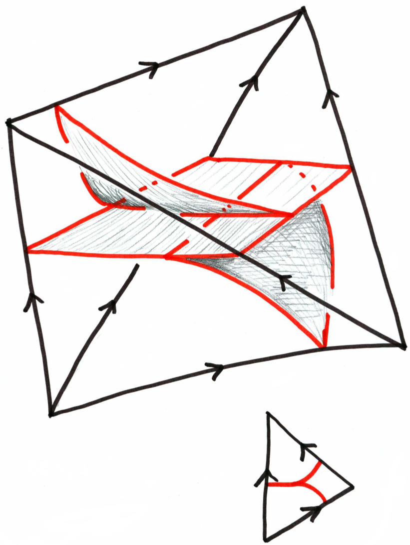

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[D. 2016] *If M has an ideal tri with a persistent lam orient, then all but at most one filling has a taut fol.*

Showing orderability:

(a) Find a taut foliation with **Euler class 0**. The action of $\pi_1(Y)$ on the universal circle then lifts to an action on $\mathbb{R}$. Works for 66,564 manifolds (22%).

(b) Find reps to $\widetilde{\mathrm{PSL}_2\mathbb{R}}$. Reps to $\mathrm{SL}_2\mathbb{R}$ are plentiful (mean 8 per mfld) but the Euler class in $H^2(Y;\mathbb{Z})$ must vanish. Works for 48,965 manifolds (16%) from 1.8 million $\mathrm{SL}_2\mathbb{R}$ reps.

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Showing not orderable: Try to order the ball in the Cayley graph of radius 3-5 in a presentation with many generators. Need fast solution to word problem: used floating-point matrix multiplication. (Discreteness is key!)

Rigorous proof:

Verified holonomy computations, a la [HIKMOT], to check that 5.8 million words are $= 1$.

Some 1Gb of "nonordering proof trees".

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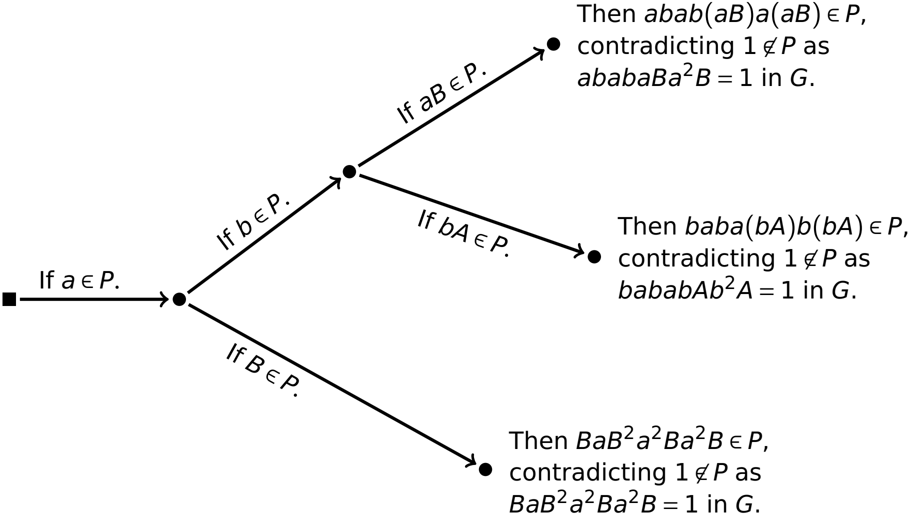

$$\pi_1(\text{Weeks}) = \left\langle a, b \;\middle|\; ababaBa^2B,\ ababAb^2Ab \right\rangle$$

The pattern: Large $|H_1(Y)|$ *increases* the odds that Y is an L-space.