Talk by Nathan Dunfield given at Univ of Warick, July 1999

Def. *A surface* S *embedded in a 3-manifold* M *is incompressible if* $\pi_1 S \to \pi_1 M$ *is injective.*

Def. *A 3-manifold is Haken if it is irreducible and contains an incompressible surface.*

Result. *Of the 246 closed hyperbolic 3-manifolds in the Hodgson-Weeks census whose volumes are less than 3, exactly 15 are Haken. These are:*

Volume	Name	Volume	Name
2.36270079	m015(8, 1)	2.70678331	m030(5,3)
2.42558538	m019(3, 4)	2.78680455	m082(1, 3)
2.60918124	m026(-5, 2)	2.81178577	m145(1, 3)
2.66674478	m036(-4, 3)	2.81251650	m070(-3, 2)
2.66674478	m040(-4, 3)	2.81251650	m069(-3, 2)
2.66674478	m140(4, 1)	2.88249439	m100(2, 3)
2.66674478	m037(4, 3)	2.97032111	m137(3, 2)
2.67947581	m034(5, 2)		

Additionally, the manifold m007(5,3) *whose volume is 2.20766623873 is also Haken.*

Motivation

Conjecture (Poincaré+) *A 3-manifold with cyclic* π_1 *is a lens space or* $S^2 \times S^1$.

Strategy introduced by Culler-Shalen:

A knot is *round* if its exterior is a solid torus.

A property Z of knots in closed 3-mflds is called *ubiquitous* if every closed, irreducible, non-Haken 3-manifold contains a knot with prop Z.

A property Z of knots is called *spiffy* if in a 3-mfld with cyclic π_1 the only knot with this prop is round.

If there is a property of knots which is both ubiquitous and spiffy, this proves Poincaré. Culler-Shalen suggested a possible property that might be both ubiquitous and spiffy. It concerns the incompressible surfaces in the exterior M of a knot K.

Let $M = \Sigma \setminus N(K)$ be the exterior of a knot in a closed 3-manifold Σ.

An isotopy class of simple closed curves in ∂M, called a slope, is determined by a class in $H_1(\partial M, \mathbb{Z})/(\pm 1)$. Choosing a nice basis, can record the slope as a number in $\mathbb{Q} \cup \infty$.

Consider a properly embedded incompressible surface in M which has torus boundary: $(S, \partial S) \hookrightarrow (M, \partial M)$.

The components of ∂S are all parallel in ∂M and so have the same slope, called the boundary slope of S.

Thm (Hatcher) *For a fixed M there are only finitely many boundary slopes*

Thus a knot has a well defined diameter of its set of boundary slopes.

Thm (Culler-Shalen) *Let* K *be a knot in a manifold with cyclic* π_1. *Then* K *is either round or the diameter of the set of boundary slopes is at least 2.*

So the following property is spiffy:

(*) The complement of K is irreducible and the diameter of the set of boundary slopes is less than 2.

Is it also ubiquitous? i.e. does every non-Haken 3-manifold have such a knot? Probably not, but some slight strengthing might well be.

I checked that in 1000's of small hyperbolic 3-manifolds there are knots with this property (short geodesics). There were a few exceptions where I was unable to find such a knot. For one of those, I was able to show that it was non-Haken. It is probably a counterexample to the ubiquitousness of (*).

Geography of volumes of orientable hyperbolic 3-manifolds:

Suppose M is a closed orientable hyperbolic 3-manifold.

Thm (Culler-Hersonsky-Shalen) *If the first betti number of* M *is at least 3 then* $\mathrm{vol}(M) > 0.946$.

Thm (Agol) *If* M *has a non-fibroid incompressible surface then* $\mathrm{vol}(M) > 2.02$.

Algorithms to decide whether a manifold is Haken

Using normal surface theory, Jaco and Oertel have given an algorithm to decide if a 3-manifold M contains an incompressible surface. In normal surface theory, you look at surfaces which meet a fixed triangulation of M in a standard way.

- If M is irreducible, any incompressible surface can be made normal.
- Finding normal surfaces is linear algebra.
- Complexity increases very rapidly in the size of the triangulation.

There are two parts to Jaco and Oertel's algorithm:

1. Enumerate a finite list of surfaces such that if there is an incompressible surface then there is one on this list.
2. Split the manifold along each of these surfaces. Apply normal surface theory again to see if any are incompressible.

Guiding Philosophy: *It's OK to do (1), but doing (2) is not a Good Idea.*

Let M be a 3-manifold with ∂M a torus. Dehn filling creates a closed manifold by gluing on a solid torus: $\left(D^2 \times S^1\right) \cup_f M$ where $f\colon \partial M \to \partial\left(D^2 \times S^1\right)$ is a homeomorphism. The homeomorphism type of M depends only on the isotopy class of $f\left(\partial D^2 \times \{\mathrm{pt}\}\right)$.

Such an isotopy class, called a slope, is determined by a class in $H_1(\partial M, \mathbb{Z})/(\pm 1)$. The Dehn filling of M so that a class α bounds a disk in the solid torus will be denoted $M(\alpha)$.

Cyclic Surgery Thm (CGLS) *Let M be a 3-manifold with torus boundary which is not Seifert fibered. Suppose $M(\alpha)$ and $M(\beta)$ have cyclic fundamental groups. Then $\Delta(\alpha, \beta) \leq 1$. In particular, there are at most 3 such slopes.*

Def. *A 3-manifold is* small *if it contains no* closed*, non-boundary parallel, incompressible surface.*

Ex. *The complement of a 2-bridge knot is small, as is a punctured torus bundle over the circle.*

Consider a properly embedded incompressible surface in a 3-manifold M with torus boundary: $(S, \partial S) \hookrightarrow (M, \partial M)$.

The components of ∂S are all parallel in ∂M and so they have the same slope in $H_1(\partial M, \mathbb{Z})/(\pm 1)$, called the boundary slope of S.

Thm (Hatcher) *For a fixed M there are only finitely many boundary slopes*

Prop. *Let M be a 3-manifold with ∂M a torus. Suppose M is small. Then if α is not the boundary slope of an incompressible surface then $M(\alpha)$ is non-Haken.*

Note: Could replace "incompressible" by "normal" because any incompressible surface can be made normal.

Algorithm (Small $\to$ non-Haken) *If M is a small manifold with ∂M a torus then it is possible to conclude that all but finitely many Dehn fillings of M are non-Haken.*

Can do this without ever deciding whether a normal surface is incompressible. Still get a finite number of exceptions because of Jaco and Sedgwick's analogue of Hatcher's theorem.

Sometimes, one can go the other direction.

Thm (Wu-CGLS) *Let* M *be an irreducible 3-manifold whose boundary is a torus. Suppose* α *and* β *are slopes such that* $M(\alpha)$ *and* $M(\beta)$ *contain no incompressible surfaces. Suppose moreover* $\Delta(\alpha, \beta) > 1$. *Then* M *is small unless there exists an incompressible surface with boundary slope* γ *such that*

$$\Delta(\alpha, \gamma) = \Delta(\alpha, \beta) = 1.$$

Leads to an easy algorithm if replace incompressible with normal.

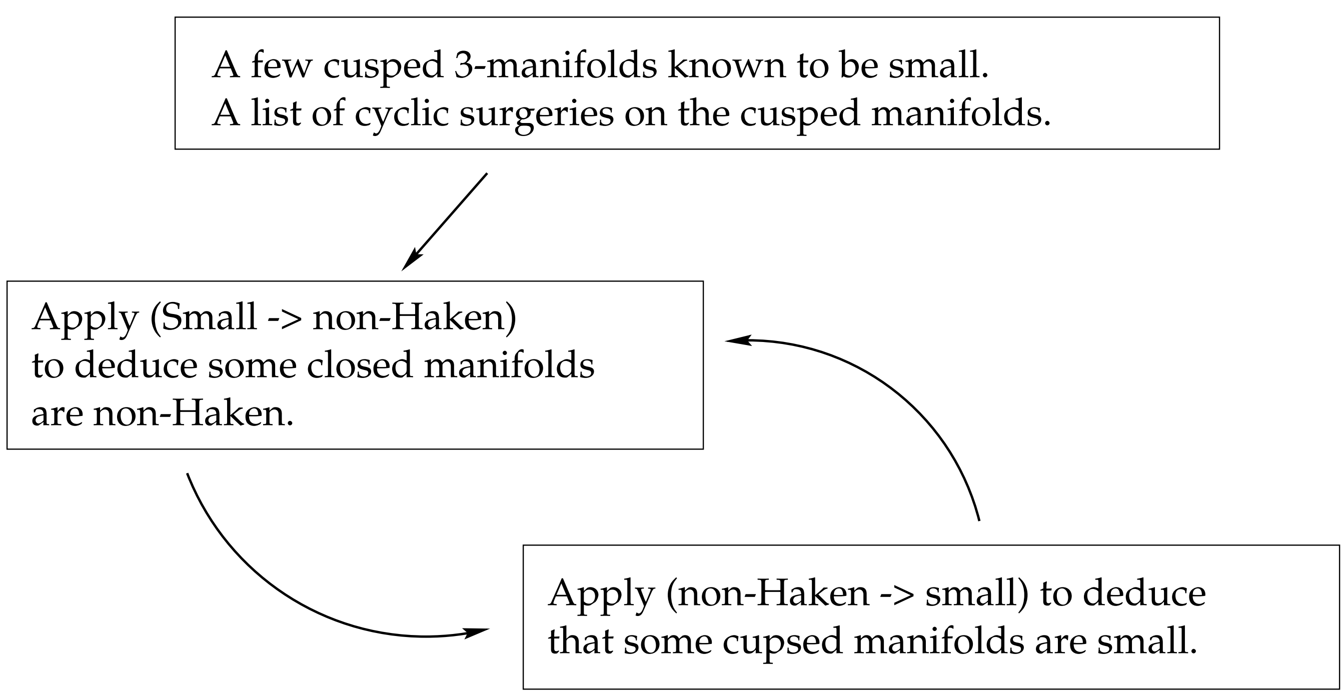
Algorithm (non-Haken $\to$ small) *If* M *is a 3-manifold with torus boundary and one knows that many Dehn fillings on* M *are non-Haken, then it may be possible to conclude that* M *is small.*

How to determine that many small volume closed hyperbolic manifolds are non-Haken

Start with:

- A census of closed orientable hyperbolic 3-manifolds (Hodgson-Weeks).
- A census of hyperbolic 3-manifolds with one cusp (Callahan-Hildebrand-Weeks).
- A list of normal slopes of each cusped manifold.

Bootstrap process:



How do decide it a closed hyperbolic 3-manifold is Haken

Thm (CGLS) *Let* M *be an irreducible 3-manifold with torus boundary and* $\dim H_1(M, \mathbb{Q}) = 1$*. If* α *is the boundary slope of an incompressible surface then either:*

1. $M(\alpha)$ *is a Haken manifold; or*
2. $M(\alpha)$ *is a connected sum of two lens spaces; or*
3. M *contains a closed incompressible surface which remains incompressible in* $M(\beta)$ *whenever*

$$\Delta(\alpha, \beta) > 1.$$

Cor. 1 *If* M *is small and* α *is a boundary slope then* $M(\alpha)$ *is Haken.*

Cor. 2 *If* α *is a boundary slope and* $M(\alpha)$ *is non-Haken then* $M(\beta)$ *is Haken for the infinitely many* β *where* $\Delta(\alpha, \beta) > 1$.

Problem is that you still need to find *incompressible* surfaces in M.

Character variety theory to the rescue (Culler-Shalen)

Can get topological information out of $PSL_2\mathbb{C}$ character varieties. Let

$$X(M) = \text{Hom}\left(\pi_1 M, PSL_2\mathbb{C}\right) \Big/ \text{conjugation},$$

an affine algebraic variety over $\mathbb{C}$. Let X_0 be an irreducible component of $X(M)$ containing the conjugacy class of a discrete faithful representation ρ_0. X_0 is an affine curve:

X_0 has a natural compactification by adding ideal points. Each ideal point has associated to it an incompressible surface. Info about the surfaces can be computed from X_0. In order to extract the information about boundary slopes it is easiest to project $X(M)$ onto $X(\partial M)$. This can be done using Gröbner bases, but this quickly becomes difficult.