

# Counting essential surfaces in 3-manifolds

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Slides posted at:  
<http://dunfield.info/slides/visgat2020.pdf>

Throughout:  $M^3$  is a cpt orient  
irreducible with every closed  
 $F^2 \subset M$  orient (e.g.  $H_2(M; \mathbb{F}_2) = 0$ ).

Closed conn embedded  $F^2 \subset M$  is  
*incompressible* when  $F \neq S^2$  and  
 $\pi_1 F \rightarrow \pi_1 M$  is injective; if  $F$  is also  
not parallel into  $\partial M$ , it is *essential*.

**Goal:** Count (closed) essential  
surfaces in  $M$ , up to isotopy.

$T^3$ : all essential surfaces are tori,  
infinitely many.

$|\pi_1 M| < \infty$ : no essential surfaces.

[Hatcher-Thurston 1985] 2-bridge  
knot exterior has no ess. surfaces.

$M^3$  is atoroidal when there are no  
 ess. tori. For atoroidal  $M$ , this is  
 always finite:

$$a_M(g) = \#\{\text{genus } g \text{ ess. surf, mod iso}\}$$

For the exterior  
 $M$  of  $11n34$ :



$g$	$a_M$	$g$	$a_M$	$g$	$a_M$
1	0	7	87	13	602
2	6	8	208	14	1,168
3	9	9	220	15	1,039
4	24	10	366	16	1,498
5	37	11	386	17	1,564
6	86	12	722	18	2,514
					...
				50	56,892
				100	444,038

$$a_M(g) = \# \left\{ \text{genus } g \text{ ess. surf, mod iso} \right\}$$

$$b_M(-n) = \# \left\{ \begin{array}{l} \text{ess. surf with } \chi = -n \\ \text{mod isotopy} \end{array} \right\}$$

For  $M = E_{11n34}$ , we show

$$b_M(-2n) = \frac{2}{3}n^3 + \frac{9}{4}n^2 + \frac{7}{3}n + \frac{7 + (-1)^n}{8}$$

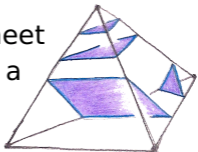
**Thm [DGR]** For atoroidal  $M^3$ , the generating function

$$\sum_{n=1}^{\infty} b_M(-2n)x^n = \frac{P(x)}{Q(x)}$$

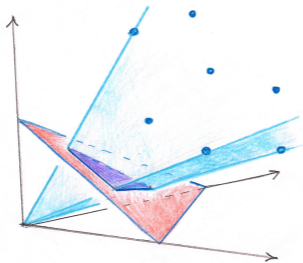
where  $P, Q \in \mathbb{Q}[x]$  and  $Q$  is a product of cyclotomics.

**Algorithm [DGR]** Can find  $P, Q$ , and isotopy reps for fixed  $\chi$ .

*Normal surfaces* meet each tetrahedra in a standard way:



and correspond to lattice points in a finite polyhedral cone  $P_T$  in  $\mathbb{R}^{7t}$  where  $t = \#T$ :



**Good:** Any essential  $F$  can be isotoped to be normal.

**Bad:** Resulting normal surface is far from unique.

weight:  $\text{wt}(F) = \#(F \cap T^1)$

lw-surface: an essential normal surface that is least weight in its isotopy class.

### **[Tollefson 90s, Oertel 80s]**

Every lw-surface lies on a lw-face  $C \subset P_T$ , one where **every** lattice point in  $C$  is a lw-surface. Isotopies between lw-surfaces can be understood.

**[Ehrhart 60s]** Counts of lattice points in rational polyhedra are quasipolynomial.

**Thm [DGR]** For atoroidal  $M^3$ , the count  $b_M(-2n)$  is quasipolynomial.

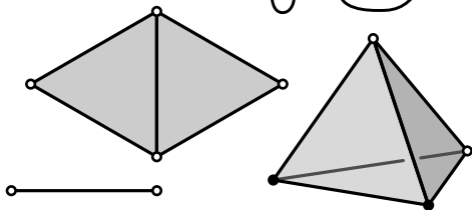
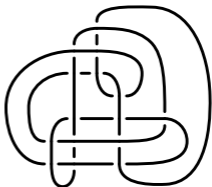
**Moral:** Ess. surf. are lattice points in the space  $\mathcal{ML}(M)$  of measured laminations [Hatcher '90s].

**Cor [DGR]** The number of ess. surfaces of  $\chi = -2n$  grows like  $n^{d-1}$  where  $d = \dim(\mathcal{ML}(M))$ .

**[Kahn-Markovic 2012]** For  $M^3$  closed hyperbolic, the number of **immersed** essential genus  $g$  surfaces grows like  $g^{2g}$ .

Computed  $\mathcal{LW}_T = \cup \{C \text{ is a lw-face}\}$   
 for 59K manifolds. Some 4K with  
 $\dim(\mathcal{LW}_T) > 1$  giving 88 distinct  $B_M$ .

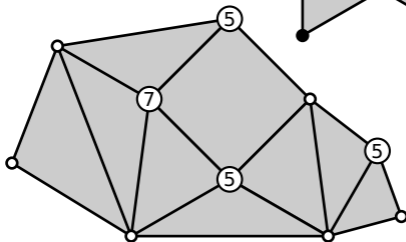
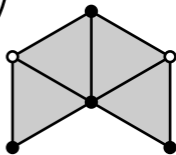
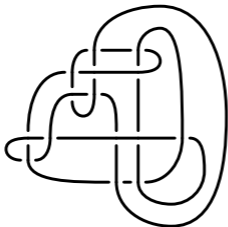
*K15n51747:*

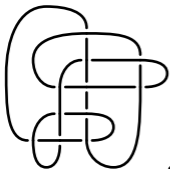


$$\frac{-3x^7 + 3x^6 + 9x^5 - 9x^4 - 9x^3 + 9x^2 + 2x}{(x-1)^4(x+1)^3}$$

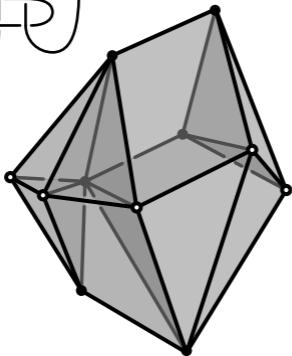


$$K15n18579: B_M(x) = \frac{-2x^6 + 5x^4 - 4x^3 - 15x^2 - 4x}{(x-1)^3(x+1)^3}$$





$K11n34$



$$B_M(x) = \frac{-x^5 + 3x^4 - 2x^3 + 2x^2 + 6x}{(x+1)(x-1)^4}$$

For  $K13n3838$ ,  $\mathcal{LW}_T$  is conn. with 44 maximal faces, all of dim 5, each with 5–9 vertex rays cor. to 48 distinct surfaces of genus 2–5. Here  $b_M(-2n)$  is:

$$\frac{7}{12}n^4 + 3n^3 + \frac{14}{3}n^2 + 3n + \frac{7 + (-1)^n}{8}$$

and  $a_M(g)$  starts 12, 34, 110, 216, 532, 708, 1558, 2018, 3462, 4176, 7314, 7876, 13204, 14256, 20778, 23404, 34820, 34832, 52226,...

## What about counting by genus?

$$a_M(g) = \#\{\text{genus } g \text{ ess. surf, mod iso}\}$$

To compute, need to decide which lattice points correspond to connected surfaces.

For the 4,330 manifolds, see 94 distinct patterns for  $a_M(g)$ .

The sequence  $a_M$  does not determine  $b_M$  or conversely.

Even for surfaces, counting connected curves only is very subtle [Mirzakhani]. We only have conjectures.

**Conj.** For  $K13n586$ , have  $a_M(2) = 2$  and  $a_M(g) = \phi(g - 1)$  for  $g > 2$ .

**Conj.** 54 of our 88 sequences  $a_M(g)$  have Möbius transform that is quasipolynomial.

**Asymptotics:**  $\bar{a}_M(g) = \sum_{k \leq g} a_M(k)$

**Conj.** Either  $a_M(g) = 0$  for all large  $g$  or there exists  $s \in \mathbb{N}$  such that  $\lim_{g \rightarrow \infty} \bar{a}_M(g)/g^s$  exists and is positive.

$\bar{a}_M$

