Lecture 4:

Previously on CGS...

[Thurston] Suppose $J$ is an ideal triangulation of $M^3$. If $z_i \in \mathbb{C}$ with $\text{Im}(z_i) > 0$ satisfy the edge and cusp eqns, then they define a complete hyperbolic structure on $M$.

Today: How can we prove such $z_i$ exist? (and that we know what they are.)

Want to understand

$$V(J) = \{ z \in \mathbb{C}^n \mid z \text{ sat the polynomial edge and cusp eqns for } J \}$$

which is an affine algebraic var defined over $\mathbb{Q}$.

Suppose $z_{\text{hyp}} \in V(J)$ sat Thurston's thm. By Mostow, only one hyp str on $M$, can use to show $z_{\text{hyp}}$ is unique. From theorem, $z_{\text{hyp}}$ is an isolated pt of $V(J)$ (local rigidity a la Calabi-Weil).

Cor: There is a number field $K \subseteq \mathbb{C}$ so that $z_{\text{hyp}} \in K^n$.

Pf: Decompose $V(J)$ into irreducible components over $\mathbb{Q}$. The pt $z_{\text{hyp}}$ is in some $0$-dim'l comp $V_0$. 

Eliminating vars to project $V_0$ onto the $i^{th}$ coor. gives a 0-dim'l variety in $C$ defined over $Q$, and so $p_i(V_0) = \text{ (roots of } f_i \in Q[x])$.

Ex: $S^3 \setminus \mathbb{H}$ \hspace{1cm} $K = Q(\alpha)$ \hspace{1cm} $\alpha = \sqrt{3}i$

$\alpha^2 + 3 = 0$

$S^3 \setminus \mathbb{H}$ \hspace{1cm} $K = Q(\beta)$ \hspace{1cm} $\beta^3 - \beta + 1$

$\beta \approx 0.6623 + 0.5622i$

$K$ is called the shape field (= trace field)

Approach 1: Use resultants/Gröbner bases/LLL ... to find $K$ and express the shapes in $Z_{hyp}$ as els of $K$.

Given $K$ as $Q[x]/f(x)$ \hspace{1cm} irreducible in $Q[x]$

and $z \in K^n$ can rigorously determine if $z \in V(J)$ since can do exact arithmetic in $K$.

[Won't work for even medium sized examples unless you get very lucky]
Approach 2: $[HIKMO] \text{ Interval Analysis.}$

$$\mathcal{II}R = \{ x = [x_0, x_1] \mid x_i \in \mathbb{Q} \}$$

View $x$ as an "enclosure" of some unknown $x \in \mathbb{R}$.

$$x + y = \{ x+y \mid x \in x, y \in y \} = [x_0+y_0, x_1+y_1]$$

Similarly for other ops: $\times, -, \div$. In practice, round endpoints but never lie: $x \times y \supseteq \{ x \times y \mid x \in x, y \in y \}$.

An interval extension $f: \mathcal{II}R \to \mathcal{II}R$ of $f: \mathbb{R} \to \mathbb{R}$ must satisfy $f(x) \supseteq \{ f(x) \mid x \in x \}$.

Ex: $\sin([1.1, 1.2]) = [0.892, 0.933]$

**Issue:** Set $d(x) = x_1 - x_0$

Have $d(x + y) = d(x) + d(y)$ $\implies$ intervals fuzz out as we do more ops.

Can say $x \neq x$ when $x \cap y = \emptyset$ but

$x = y$ is not allowed.

**Point:** The proof of the Inverse Fn. Thm is effective and can be used to show there is a point in $V(J)$ in some small $\varepsilon \in (\mathcal{II}C)$.
Thm: Suppose $f: \mathbb{R} \to \mathbb{R}$ is differentiable. Given $x \in [x_0, x_1] \subseteq \mathbb{R}$ with $0 \neq f'(x)$ define for $\tilde{x} \in [x_0, x_1]$ the quantity

$$N(\tilde{x}, x) = \tilde{x} - \frac{f(\tilde{x})}{f'(x)} \in \mathbb{R}.$$ 

If $N(\tilde{x}, x) \leq x$ then there exists a unique root of $f$ in $x$.

Proof by picture: Assume $f(\tilde{x}) > 0$ and $f'(\tilde{x}) > 0$.

Point: Slope of graph of $f$ on $x$ is constrained by $f(x)$. Thus, $f$ must have a root in $N(\tilde{x}, x)$. 
There exist analogs for $F: \mathbb{R}^n \to \mathbb{R}^n$ or $C^n \to C^n$ called "interval Newton's method" and "Krawczyk's test." Rely on Brouwer's Fixed Point Theorem.

[Neumann-Zagier] $V(J)$ can be cut out by # tet equations.

[Demo: SnapPy and SageMath are friends]

The End